

# Measurable Versions of the Lovász Local Lemma and Measurable Graph Colorings

Anton Bernshteyn\*

University of Illinois at Urbana-Champaign

## Abstract

In this paper we investigate the extent to which the Lovász Local Lemma (an important tool in probabilistic combinatorics) can be adapted for the measurable setting. In most applications, the Lovász Local Lemma is used to produce a function  $f: X \rightarrow Y$  with certain properties, where  $X$  is some underlying combinatorial structure and  $Y$  is a (usually finite) set. Can this function  $f$  be chosen to be Borel or  $\mu$ -measurable for some probability Borel measure  $\mu$  on  $X$  (assuming that  $X$  is a standard Borel space)? In the positive direction, we prove that if the set of constraints put on  $f$  is, in a certain sense, “locally finite,” then there is always a Borel choice for  $f$  that is “ $\varepsilon$ -close” to satisfying these constraints, for any  $\varepsilon > 0$ . Moreover, if the combinatorial structure on  $X$  is “induced” by the  $[0; 1]$ -shift action of a countable group  $\Gamma$ , then, even without any local finiteness assumptions, there is a Borel choice for  $f$  which satisfies the constraints on an invariant conull set (i.e., with  $\varepsilon = 0$ ). On the other hand, at least for amenable groups, the last statement is sharp: a probability measure-preserving action of a countably infinite amenable group satisfies the measurable version of the Lovász Local Lemma if and only if it admits a factor map to the  $[0; 1]$ -shift action. A direct corollary of our results is an upper bound on the measurable chromatic number of the graph  $G_n$  generated by the shift action of the free group  $\mathbb{F}_n$ , that is asymptotically tight up to a factor of at most 2 (which answers a question of Lyons and Nazarov).

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	Graph colorings in the Borel and measurable settings . . . . .	2
1.2	The Lovász Local Lemma and its applications . . . . .	7
1.3	Overview of our main results and the structure of the paper . . . . .	11
1.4	General notation, conventions, and preliminary results . . . . .	13
<b>2</b>	<b>Moser–Tardos theory</b>	<b>15</b>
2.1	Moser–Tardos theory in the Borel setting . . . . .	21
<b>3</b>	<b>Hereditarily finite sets</b>	<b>23</b>

---

\*This research is supported by the Illinois Distinguished Fellowship.

<b>4</b>	<b>The approximate Borel LLL</b>	<b>24</b>
4.1	Proof of Theorem 4.1 . . . . .	24
<b>5</b>	<b>The LLL for probability measure-preserving group actions</b>	<b>27</b>
5.1	Definitions and the statement of the theorem . . . . .	27
5.2	Proof of Theorem 5.4 . . . . .	30
<b>6</b>	<b>The converse of Theorem 5.4 for actions of amenable groups</b>	<b>42</b>
6.1	Proofs of Theorems 6.1 and 6.1' . . . . .	44
<b>Appendix A Proof of Theorem 2.5</b>		<b>58</b>
<b>Appendix B Theorems of Kim, Johansson, and Kahn</b>		<b>62</b>

# 1 Introduction

## 1.1 Graph colorings in the Borel and measurable settings

In this paper we investigate the extent to which some classical results in finite combinatorics can be transferred to the measurable setting. Our main object of study will be the so-called Lovász Local Lemma, which we will discuss in some detail in the next subsection. Here we give a “preview” of particular applications that our general techniques can provide.

First, let us recall some definitions<sup>1</sup>. A *graph*  $G$  with vertex set  $X$  (or a graph on  $X$ ) is a symmetric irreflexive binary relation on  $X$ . In particular, unless stated otherwise, graphs in this paper are assumed to be undirected and simple. We call two vertices  $x, y \in X$  *adjacent* in  $G$  if  $xGy$ . A subset  $X' \subseteq X$  is  *$G$ -invariant* if no vertex in  $X'$  is adjacent to a vertex in  $X \setminus X'$ . A *connected component* of  $G$  is an inclusion-minimal nonempty  $G$ -invariant subset of  $X$ . If  $X' \subseteq X$ , then  $G|X' := G \cap (X')^2$  denotes the *subgraph* of  $G$  *induced*<sup>2</sup> by  $X'$  (or, equivalently, the *restriction* of  $G$  to  $X'$ ). The *degree* of a vertex  $x \in X$  (notation:  $\deg_G(x)$  or simply  $\deg(x)$ ) is the cardinality of the set  $G_x := \{y \in X : xGy\}$ . The *maximum degree* of  $G$  (notation:  $\Delta(G)$ ) is the supremum of  $\deg(x)$  over all  $x \in X$ . A graph  $G$  is said to be *locally countable* if  $\Delta(G) \leq \aleph_0$ ;  $G$  is said to be *locally finite* if  $\deg(x) < \aleph_0$  for all  $x \in X$ . The *girth* of  $G$  (notation:  $g(G)$ ) is the length of the shortest cycle in  $G$  (if  $G$  is acyclic,  $g(G) = \infty$  by definition). A *proper (vertex) coloring* of  $G$  is a map  $f: X \rightarrow Y$ , where  $Y$  is a set of *colors*, such that for any two adjacent vertices  $x, y \in X$ ,  $f(x) \neq f(y)$ . The *chromatic number* of  $G$  (notation:  $\chi(G)$ ) is the smallest cardinality of a set  $Y$  such that  $G$  admits a proper coloring  $f: X \rightarrow Y$ .

We will be interested in the properties of Borel graphs; see [21] for a comprehensive survey of the topic. Suppose that  $X$  is a standard Borel space. A graph  $G$  on  $X$  is said to be *Borel*

<sup>1</sup>Graph-theoretic notation and terminology used in descriptive set theory deviate somewhat from what is generally accepted as standard in finite combinatorics. The notation used in this paper is more aligned with the descriptive set-theoretic one. For instance, we identify a graph  $G$  with its edge set; the notation  $E(G)$ , common in finite combinatorics, would be in conflict with  $E_G$ —the equivalence relation whose classes are the connected components of  $G$ .

<sup>2</sup>In finite combinatorics the standard notation for this concept is  $G[X']$ . However, such notation would be in conflict with  $[X']_{E_G}$ , the  $G$ -saturation of  $X'$ .

if it is Borel as a subset of  $X^2$ . An important source of Borel graphs are Borel group actions. Let  $\Gamma$  be a countable group acting by Borel automorphisms on a standard Borel space  $X$  (in this paper we only consider left group actions). Denote this action by  $a: \Gamma \curvearrowright X$ . Let  $S \subseteq \Gamma$  be a generating set and define the graph  $G(a, S)$  on  $X$  via

$$xG(a, S)y :\iff x \neq y \text{ and } \exists \gamma \in S \cup S^{-1}(\gamma \cdot x = y).$$

Then  $G(a, S)$  is locally countable and Borel.

For a Borel graph  $G$  on  $X$ , its *Borel chromatic number* (notation:  $\chi_B(G)$ ) is the smallest cardinality of a standard Borel space  $Y$  such that  $G$  admits a Borel proper coloring  $f: X \rightarrow Y$ . Borel chromatic numbers were first introduced and systematically studied by Kechris, Solecki, and Todorćević [23]. Clearly,  $\chi(G) \leq \chi_B(G)$ . One of the starting points of Borel combinatorics is the observation that this inequality can be strict. In fact, Kechris, Solecki, and Todorćević [23, Example 3.1] gave an example of an *acyclic* locally countable Borel graph  $G$  such that  $\chi_B(G) = 2^{\aleph_0}$  (note that if  $G$  is acyclic, then  $\chi(G) \leq 2$ ). However, one can show that if the maximum degree of  $G$  is finite, then  $\chi_B(G) \leq \Delta(G) + 1$ —similarly to the finite case (where this statement is known as a special case of the Szekeres–Wilf theorem [36]).

The bound  $\chi(G) \leq \Delta(G) + 1$  is rather weak; in fact, Brooks’s theorem in finite combinatorics asserts that  $\chi(G) \leq \Delta(G)$  for all  $G$  apart from a few natural exceptions [4, Theorem 14.4]. However, as it turned out, there is no hope for any result along those lines in the Borel setting: Marks [30, Theorem 1.3] showed that the Borel chromatic number of an acyclic Borel graph  $G$  with maximum degree  $d \in \mathbb{N}$  can attain the value  $d + 1$  (and, in fact, *any* value between 2 and  $d + 1$ ).

Marks’s results indicate that the Borelness condition is too restrictive to allow any interesting analogs of classical coloring results. It is reasonable, therefore, to try asking for somewhat less. For instance, we can only require that “most” of the graph should be colored, in an appropriate sense of the word “most.” Natural candidates for such a notion of largeness are Baire category and measure. If  $\tau$  is a Polish topology on  $X$  that is compatible with the Borel structure on  $X$ , then the  $\tau$ -*Baire-measurable chromatic number* of  $G$  is defined as follows:

$$\chi_\tau(G) := \min\{\chi_B(G|X') : X' \text{ is a } \tau\text{-comeager } G\text{-invariant Borel subset of } X\}.$$

Similarly, if  $\mu$  is a probability Borel measure on  $X$ , then the  $\mu$ -*measurable chromatic number* of  $G$  is defined to be

$$\chi_\mu(G) := \min\{\chi_B(G|X') : X' \text{ is a } \mu\text{-conull } G\text{-invariant Borel subset of } X\}.$$

Just like  $\chi_B(G)$ ,  $\chi_\tau(G)$  and  $\chi_\mu(G)$  can both exceed  $\chi(G)$ , even for locally finite acyclic graphs. A simple example is given by the graph  $G := G(a, \{1\})$ , where  $a: \mathbb{Z} \curvearrowright \mathbb{S}^1$  is an irrational rotation action of  $\mathbb{Z}$  on the unit circle  $\mathbb{S}^1$ . Note that each connected component of  $G$  is a bi-infinite path, so  $G$  is acyclic. However, an easy ergodicity argument reveals that both  $\chi_\tau(G)$  and  $\chi_\mu(G)$  are greater than 2, where  $\tau$  is the usual topology on  $\mathbb{S}^1$  and  $\mu$  is the Lebesgue probability measure on  $\mathbb{S}^1$ . In fact, since  $\Delta(G) = 2$ ,  $\chi_\tau(G) = \chi_\mu(G) = \chi_B(G) = 3$ .

Nevertheless, Conley and Miller [11, Theorem B] showed that  $\chi_\tau(G)$  cannot differ from  $\chi(G)$  too much; namely, they proved that for a locally finite Borel graph  $G$  on a standard

Borel space  $X$ , if  $\chi(G)$  is finite, then  $\chi_\tau(G) \leq 2\chi(G) - 1$  with respect to any compatible Polish topology  $\tau$  on  $X$ . In particular, if  $G$  is acyclic (or, more generally,  $\chi(G) \leq 2$ ), then  $\chi_\tau(G) \leq 3$ .

Our main focus in this paper will be on  $\mu$ -measurable chromatic numbers (and  $\mu$ -measurable analogs of other combinatorial parameters). Here the situation is more intriguing than with Baire-measurable chromatic numbers. Conley, Marks, and Tucker-Drob [10, Theorem 1.2] recently proved a  $\mu$ -measurable analog of Brooks's theorem for graphs with maximum degree at least 3 (the example of an irrational rotation action shows that Brooks's theorem for graphs with maximum degree 2 does not hold in the measurable setting). In particular,  $\chi_\mu(G)$  can be *strictly less* than  $\chi_B(G)$ .

On the other hand, in contrast to  $\chi_\tau(G)$ ,  $\chi_\mu(G)$  *cannot* be bounded above by any function of  $\chi(G)$ . An important class of examples where the difference between  $\chi_\mu(G)$  and  $\chi(G)$  gets arbitrarily large comes from shift actions of free groups. For a countable group  $\Gamma$  and a set  $Y$ , the *shift action*<sup>3</sup> of  $\Gamma$  on  $Y^\Gamma$  (or the  *$Y$ -shift action* of  $\Gamma$ ) is defined as follows: For all  $\gamma, \delta \in \Gamma$  and  $x \in Y^\Gamma$ , let

$$(\gamma \cdot x)(\delta) := x(\delta\gamma).$$

Let  $S$  be a finite set and let  $\mathbb{F}(S)$  denote the free group over  $S$ . Let  $a: \mathbb{F}(S) \curvearrowright [0; 1]^{\mathbb{F}(S)}$  be the shift action of  $\mathbb{F}(S)$  on  $[0; 1]^{\mathbb{F}(S)}$  and let  $G := G(a, S)$ . Let  $\lambda$  denote the Lebesgue measure on  $[0; 1]$  (we will use this notation throughout). Then, off of a  $\lambda^{\mathbb{F}(S)}$ -null set, the action  $a$  is free, so each connected component of  $G$  is an infinite  $2|S|$ -regular tree, and hence it is 2-colorable. However, as Lyons and Nazarov [29] pointed out, a result of Frieze and Łuczak [16] implies that  $\chi_{\lambda^{\mathbb{F}(S)}}(G) \geq |S|/\ln(2|S|)$  for sufficiently large  $|S|$  (see also [21, Theorem 5.44], where this lower bound is proved for arbitrary  $S$ ). In particular,  $\chi_{\lambda^{\mathbb{F}(S)}}(G) \rightarrow \infty$  as  $|S| \rightarrow \infty$ . Note that the group  $\mathbb{F}(S)$  for  $|S| \geq 2$  is nonamenable; in fact, Conley and Kechris [9] mention that there are no known examples of graphs  $G$  induced by probability measure-preserving actions of amenable groups such that  $\chi_\mu(G) > \chi(G) + 1$  (see [21, Problem 5.19]).

Note that the best known *upper* bound on  $\chi_{\lambda^{\mathbb{F}(S)}}(G)$  is  $2|S|$  (given by the measurable Brooks's theorem), so the orders of magnitude of the lower and upper bounds are different. Lyons and Nazarov [29] asked what the correct value of  $\chi_{\lambda^{\mathbb{F}(S)}}(G)$  should be. As an immediate corollary of one of our main results (namely Theorem 5.4), we can show that  $|S|/\ln(|S|)$  is the right order. In fact, we have the following general theorem:

**Theorem 1.1.** *Let  $\Gamma$  be a countable group with a finite generating set  $S \subseteq \Gamma$ ; denote  $d := |S \cup S^{-1}|$ . Let  $a: \Gamma \curvearrowright (X, \mu)$  be a measure-preserving action of  $\Gamma$  on a standard probability space  $(X, \mu)$  and let  $G := G(a, S)$ . Suppose that there exists a factor map  $\pi: (X, \mu) \rightarrow ([0; 1]^\Gamma, \lambda^\Gamma)$  to the shift action of  $\Gamma$  on  $[0; 1]^\Gamma$ . If  $g(G) \geq 4$ , then*

$$\chi_\mu(G) = O\left(\frac{d}{\ln d}\right).$$

Moreover, if  $g(G) \geq 5$ , then

$$\chi_\mu(G) \leq (1 + o(1))\frac{d}{\ln d}.$$

---

<sup>3</sup>There are two ways to define the shift action: either by multiplying on the right, or by multiplying on the left with the inverse. The latter one is more common; however, we prefer the former one, especially since it will turn out to be more convenient when we get to §5.2.3.

**Corollary 1.2.** *Let  $S$  be a finite set of size  $d$ , let  $a: \mathbb{F}(S) \curvearrowright [0; 1]^{\mathbb{F}(S)}$  be the  $[0; 1]$ -shift action of the free group  $\mathbb{F}(S)$ , and let  $G := G(a, S)$ . Then*

$$(1 - o(1)) \frac{d}{\ln d} \leq \chi_{\lambda^r}(G) \leq (2 + o(1)) \frac{d}{\ln d}. \quad (1.1)$$

Note that, by a result of Bowen [6, Theorem 1.1], any two nontrivial<sup>4</sup> shift actions of  $\mathbb{F}(S)$ , where  $|S| \geq 2$ , admit factor maps into each other, so (1.1) holds for any such action as well.

Another extensively studied graph parameter is the so-called chromatic index of a graph. Let  $G$  be a graph with vertex set  $X$ . An *edge coloring* of  $G$  is a map  $f: G \rightarrow Y$  such that for all  $x, y \in X$  with  $xGy$ ,  $f(x, y) = f(y, x)$ . An edge coloring  $f$  is *proper* if for all  $x, y, z \in X$  with  $xGy$ ,  $yGz$ , and  $x \neq z$ ,  $f(x, y) \neq f(y, z)$ . The *chromatic index* of  $G$  (notation:  $\chi'(G)$ ) is the smallest cardinality of a set  $Y$  such that  $G$  admits a proper edge coloring  $f: G \rightarrow Y$ . Clearly,  $\chi'(G) \geq \Delta(G)$ , since all the edges incident to a vertex  $x \in X$  have to receive distinct colors. A celebrated theorem of Vizing [4, Theorem 17.4] asserts that this bound is almost tight; namely, for a finite graph  $G$ ,  $\chi'(G) \leq \Delta(G) + 1$ .

Naturally, for a Borel graph  $G$  on a standard Borel space  $X$ , its *Borel chromatic index* (notation:  $\chi'_B(G)$ ) is defined to be the smallest cardinality of a standard Borel space  $Y$  such that  $G$  admits a Borel proper edge coloring  $f: G \rightarrow Y$  (where  $G$  inherits its Borel structure from  $X^2$ ). Clearly,  $\chi'(G) \leq \chi'_B(G)$ . In the aforementioned paper [30, Theorem 1.4], Marks showed that the Borel chromatic index of an acyclic Borel graph  $G$  with maximum degree  $d \in \mathbb{N}$  can be as large as  $2d - 1$  (and this bound is tight—finding a proper edge coloring of a graph with maximum degree  $d$  is equivalent to finding a proper *vertex* coloring of an auxiliary graph with maximum degree  $2d - 2$ ).

One can define the  $\mu$ -measurable chromatic index of a Borel graph  $G$  by analogy with its  $\mu$ -measurable chromatic number; namely,

$$\chi'_\mu(G) := \min\{\chi'_B(G|X') : X' \text{ is a } \mu\text{-conull } G\text{-invariant Borel subset of } X\}.$$

Csóka, Lippner, and Pikhurko [13, Theorem 1.4] proved that Vizing's theorem holds measurably for bipartite graphs and that  $\chi_\mu(G) \leq \Delta(G) + o(\Delta(G))$  in general, provided that the measure  $\mu$  is  $G$ -invariant. Theorem 5.4 gives a different proof of the second part of this result for graphs induced by shift actions (with a slightly worse lower order term); moreover, it implies the following “list version”:

**Theorem 1.3.** *For every  $d \in \mathbb{N}$ , there exists  $k = d + o(d)$  such that the following holds. Let  $\Gamma$  be a countable group with a finite generating set  $S \subseteq \Gamma$  such that  $|S \cup S^{-1}| = d$ . For each  $\gamma \in S \cup S^{-1}$ , let  $L(\gamma)$  be a finite set such that for all  $\gamma \in S \cup S^{-1}$ ,  $L(\gamma) = L(\gamma^{-1})$  and  $|L(\gamma)| \geq k$ . Let  $a: \Gamma \curvearrowright (X, \mu)$  be a measure-preserving action of  $\Gamma$  on a standard probability space  $(X, \mu)$  and let  $G := G(a, S)$ . Suppose that there exists a factor map  $\pi: (X, \mu) \rightarrow ([0; 1]^\Gamma, \lambda^\Gamma)$  to the shift action of  $\Gamma$  on  $[0; 1]^\Gamma$ . Then there exists a  $\Gamma$ -invariant  $\mu$ -conull Borel subset  $X' \subseteq X$  and a Borel proper edge coloring  $f$  of  $G|X'$  such that for all  $x \in X'$ ,  $f(x, \gamma \cdot x) \in L(\gamma)$ .*

One can further relax the conditions on a coloring to allow a small (but positive) margin of error. Let  $G$  be a graph with vertex set  $X$ . For a map  $f: X \rightarrow Y$ , define the *defect*

---

<sup>4</sup>A probability measure  $\nu$  is said to be *nontrivial* if it is not concentrated on a single point.

set  $D_f \subseteq X$  by

$$x \in D_f : \Longleftrightarrow \exists y \in G_x(f(x) = f(y)).$$

In other words, a vertex  $x$  belongs to  $D_f$  if and only if it shares a color with at least one of its neighbors. If the graph  $G$  is Borel, then a Borel map  $f: X \rightarrow Y$  is a  $(\mu, \varepsilon)$ -*approximately proper Borel coloring* of  $G$  if  $\mu(D_f) \leq \varepsilon$ .<sup>5</sup> The  $\mu$ -*approximate chromatic number* of  $G$  (notation:  $\text{ap}\chi_\mu(G)$ ) is the smallest cardinality of a standard Borel space  $Y$  such that for every positive  $\varepsilon$ , there is a  $(\mu, \varepsilon)$ -approximately proper Borel coloring  $f: X \rightarrow Y$  of  $G$ . Approximate chromatic numbers were studied extensively by Conley and Kechris [9].<sup>6</sup> In particular, Conley and Kechris proved that if  $G$  is induced by a measure-preserving action of a countable *amenable* group, then its  $\mu$ -approximate chromatic number is essentially determined by the ordinary chromatic number; more precisely, for such  $G$ ,

$$\text{ap}\chi_\mu(G) = \min\{\chi(G|X') : X' \text{ is a } \mu\text{-conull } G\text{-invariant Borel subset of } X\}.$$

However, the lower bound  $\text{ap}\chi_{\lambda_{\mathbb{F}(S)}}(G(a, S)) \geq |S|/\ln(2|S|)$ , where  $a: \mathbb{F}(S) \curvearrowright [0; 1]^{\mathbb{F}(S)}$  is the shift action of the free group  $\mathbb{F}(S)$  over a finite set  $S$ , still holds.

Similarly, for an edge coloring  $f: G \rightarrow Y$ , let  $D'_f \subseteq X$  be given by

$$x \in D'_f : \Longleftrightarrow \begin{array}{l} \exists y \in G_x \exists z \in G_y (z \neq x \text{ and } f(x, y) = f(y, z)); \text{ or} \\ \exists y \in G_x \exists z \in G_x (z \neq y \text{ and } f(x, y) = f(x, z)). \end{array}$$

In other words,  $x \in D'_f$  if and only if it is incident to an edge that shares an endpoint with another edge of the same color. The definition of the  $(\mu, \varepsilon)$ -*approximate chromatic index* of a Borel graph  $G$  is analogous to that of  $\text{ap}\chi_\mu(G)$ . As a corollary of our other general result (namely Theorem 4.1), Theorems 1.1 and 1.3 can be generalized to *arbitrary* Borel graphs in the context of approximate colorings.

**Theorem 1.4.** *Let  $G$  be a Borel graph on a standard Borel space  $X$  and suppose that  $\Delta(G) = d \in \mathbb{N}$ . Let  $\mu$  be a probability Borel measure on  $X$ . If  $g(G) \geq 4$ , then*

$$\text{ap}\chi_\mu(G) = O\left(\frac{d}{\ln d}\right).$$

Moreover, if  $g(G) \geq 5$ , then

$$\text{ap}\chi_\mu(G) \leq (1 + o(1)) \frac{d}{\ln d}.$$

**Theorem 1.5.** *Let  $G$  be a Borel graph on a standard Borel space  $X$  and suppose that  $\Delta(G) = d \in \mathbb{N}$ . Let  $\mu$  be a probability Borel measure on  $X$ . Then*

$$\text{ap}\chi'_\mu(G) = d + o(d).$$

---

<sup>5</sup>Note that the set  $D_f$  is analytic (and hence universally measurable), so this definition makes sense. Moreover, if  $G$  is locally countable, then  $D_f$  is actually Borel.

<sup>6</sup>The definition used in [9] is slightly weaker than the one stated above; namely, Conley and Kechris only ask for a Borel set  $X' \subseteq X$  of measure at least  $1 - \varepsilon$  such that  $f|_{X'}$  is a proper coloring of  $G|_{X'}$ . However, the two definitions are easily seen to be equivalent when  $\Delta(G)$  is finite and  $\mu$  is  $G$ -invariant.

## 1.2 The Lovász Local Lemma and its applications

The Lovász Local Lemma (the LLL for short) is an immensely useful combinatorial tool developed by Erdős and Lovász in their seminal paper [14]. The Lovász Local Lemma is usually stated in probabilistic terms. However, it will be more convenient for us to use the following (equivalent) measure-theoretic version.

**Theorem 1.6 (The Lovász Local Lemma; Erdős–Lovász [14]; see also [3, Lemma 5.1.1]).** *Let  $X$  be a standard Borel space and let  $\mu$  be a probability Borel measure on  $X$ . Let  $\mathcal{B}$  be a finite family of Borel (or, more generally,  $\mu$ -measurable) subsets of  $X$ . For each  $B \in \mathcal{B}$ , let  $\mathcal{N}(B)$  be a subset of  $\mathcal{B}$  such that  $B \in \mathcal{N}(B)$  and  $B$  is independent from the algebra generated by  $\mathcal{B} \setminus \mathcal{N}(B)$ . Suppose that there is a function  $p: \mathcal{B} \rightarrow [0; 1)$  such that for every  $B \in \mathcal{B}$ , we have*

$$\mu(B) \leq \frac{p(B)}{1 - p(B)} \prod_{B' \in \mathcal{N}(B)} (1 - p(B')). \quad (1.2)$$

*Then  $\mu(X \setminus \bigcup_{B \in \mathcal{B}} B) \geq \prod_{B \in \mathcal{B}} (1 - p(B)) > 0$ .*

The LLL is commonly used in order to prove combinatorial existence results. In a typical application of the LLL, one attempts to find a map (a “coloring”)  $f: X \rightarrow Y$ , where  $X$  is a finite set, subject to certain “local” constraints. The strategy is to put a suitable probability measure on  $Y$  and choose  $f$  randomly from  $Y^X$  according to the product probability measure. If the requirements of the LLL are fulfilled, then the set of acceptable  $f \in Y^X$  has positive measure and hence is nonempty. Since for each  $x \in X$ ,  $f(x)$  in such situation can be viewed as a random variable with values in  $Y$  and these variables are mutually independent, this framework is sometimes referred to as the *variable version* of the LLL (the name is due to Kolipaka and Szegedy [25]).

Let us now state the variable version of the LLL formally. For sets  $X$  and  $Y$ ,  $[X]^{<\infty}$  denotes the set of all nonempty finite subsets of  $X$  and  $[X]_Y^{<\infty}$  denotes the set of all functions  $f: S \rightarrow Y$ , where  $S \in [X]^{<\infty}$ . For  $B \subseteq [X]_Y^{<\infty}$  and  $S \in [X]^{<\infty}$ , let  $B_S := B \cap Y^S$ . An *instance (of the LLL)* over a set  $X$  is a subset  $B \subseteq [X]_{[0;1]}^{<\infty}$  such that for each  $S \in [X]^{<\infty}$ , the set  $B_S \subseteq [0; 1]^S$  is Borel. (Note that so far we have only assumed that  $X$  is a set with no additional structure on it; however,  $B_S$  is a subset of the standard Borel space  $[0; 1]^S$ , so the above definition makes sense.) The *domain* of an instance  $B$  is the set

$$\text{dom}(B) := \{S \in [X]^{<\infty} : B_S \neq \emptyset\}.$$

We say that  $B$  is *locally countable* if for each  $x \in X$ , the set

$$\text{dom}_x(B) := \{S \in \text{dom}(B) : x \in S\}$$

is countable. An instance  $B$  is said to be *correct* if it is locally countable and there is a function  $p: \text{dom}(B) \rightarrow [0; 1)$  such that for all  $S \in \text{dom}(B)$ ,

$$\lambda^S(B_S) \leq \frac{p(S)}{1 - p(S)} \prod_{\substack{S' \in \text{dom}(B): \\ S' \cap S \neq \emptyset}} (1 - p(S')), \quad (1.3)$$

where  $\lambda$  denotes the Lebesgue measure on  $[0; 1]$ . (The local countability of  $B$  ensures that the product on the right-hand side of (1.3) makes sense.) Note that if  $p$  satisfies (1.3), then, without loss of generality, we may assume that  $p(S) = 0$  whenever  $\lambda^S(B_S) = 0$ . A *solution* for an instance  $B$  is a function  $f: X \rightarrow [0; 1]$  such that  $f|_S \notin B_S$  for all  $S \in \text{dom}(B)$ .

**Corollary 1.7 (The variable version of the LLL—finite case).** *Let  $X$  be a finite set and let  $B$  be a correct instance over  $X$ . Then there exists a solution  $f$  for  $B$ .*

*Proof.* For each  $S \in \text{dom}(B)$ , we will abuse notation and denote by  $B_S$  the set of all maps  $f: X \rightarrow [0; 1]$  with  $f|_S \in B$ . Let  $\mathcal{B} := \{B_S : S \in \text{dom}(B)\}$ . For each  $S \in \text{dom}(B)$ , define

$$\mathcal{N}(B_S) := \{B_{S'} : S' \in \text{dom}(B) \text{ and } S' \cap S \neq \emptyset\}.$$

Let  $p: \text{dom}(B) \rightarrow [0; 1]$  be a function witnessing the correctness of  $B$ . It is easy to see that, after setting  $p(B_S) := p(S)$ , the conditions of the LLL are fulfilled, so

$$\lambda^X \left( [0; 1]^X \setminus \bigcup_{S \in \text{dom}(B)} B_S \right) > 0.$$

In particular, the set  $[0; 1]^X \setminus \bigcup_{S \in \text{dom}(B)} B_S$  is nonempty, as desired. ■

Kun [26, Lemma 13] observed that, in fact, the statement of Corollary 1.7 also holds for *arbitrary*  $X$ :

**Theorem 1.8 (The variable version of the LLL—general case; Kun [26, Lemma 13]).** *Let  $X$  be a set and let  $B$  be a correct instance over  $X$ . Then there exists a solution  $f$  for  $B$ .*

The main ingredient in the proof of Theorem 1.8 is the effective approach to the LLL developed by Moser and Tardos [33]. Since it will also play a key role in our investigation, we will present the main tools of the Moser–Tardos theory in Section 2.<sup>7</sup>

*Remark 1.9.* The definition of an instance can be naturally generalized to include subsets of  $[X]_Y^{<\infty}$  for standard probability spaces  $(Y, \mu)$  other than  $([0; 1], \lambda)$ . For example, in combinatorial applications the space  $Y$  is usually taken to be discrete. However, any standard probability space can be “simulated” by  $([0; 1], \lambda)$ , i.e., for any standard probability space  $(Y, \mu)$ , there exists a Borel map  $\varphi: [0; 1] \rightarrow Y$  such that  $\varphi_*(\lambda) = \mu$ ; and, for the purposes of applying the LLL, a subset  $B \subseteq [X]_Y^{<\infty}$  can be replaced by its “pullback”  $\varphi^*(B) \subseteq [X]_{[0; 1]}^{<\infty}$  defined via

$$f \in \varphi^*(B) :\iff \varphi \circ f \in B.$$

Therefore, we do not lose in generality when only considering subsets of  $[X]_{[0; 1]}^{<\infty}$  (and it makes our notation simpler).

---

<sup>7</sup>The statement of Theorem 1.8 for instances  $B$  appearing in concrete combinatorial applications can often be derived from Corollary 1.7 through a compactness argument, since the sets  $B_S$  for  $S \in [X]^{<\infty}$  can typically be arranged to be open. However, a different argument is required in general.



For example, let  $\mathcal{H}$  be a  $k$ -uniform hypergraph with vertex set  $X$ , i.e., a collection of  $k$ -element subsets of  $X$ , called the *edges* of  $\mathcal{H}$ . A *proper 2-coloring* of  $\mathcal{H}$  is a map  $f: X \rightarrow \{0, 1\}$  such that every edge  $H \in \mathcal{H}$  contains vertices of both colors. For  $f \in [X]_{\{0,1\}}^{\leq \infty}$ , let

$$f \in B :\iff \text{dom}(f) \in \mathcal{H} \text{ and } f \text{ is constant on its domain.}$$

As explained in Remark 1.9,  $B$  can be viewed as an instance over  $X$ . Note that proper 2-colorings of  $\mathcal{H}$  are exactly the solutions for  $B$ . Therefore, to show that  $\mathcal{H}$  is 2-colorable, it suffices to prove that  $B$  is correct. In this case it is straightforward to check when (1.3) is satisfied. After a small calculation, one recovers the following theorem due to Erdős and Lovász (which was historically the first application of the LLL).

**Theorem 1.10** (Erdős–Lovász [14]). *Let  $\mathcal{H}$  be a  $k$ -uniform hypergraph and suppose that every edge in  $\mathcal{H}$  intersects at most  $d$  other edges. If  $e(d+1) \leq 2^{k-1}$ , then  $\mathcal{H}$  is 2-colorable.<sup>8</sup>*

To illustrate the types of results one can get using the variable version of the LLL, we describe a few other applications below.

**Kim’s and Johansson’s theorems** Suppose that  $G$  is a graph with “large” girth. Can we show that  $\chi(G)$  is much smaller than  $\Delta(G)$ , the bound given by Brooks’s theorem? It is well-known that there exist  $d$ -regular graphs with arbitrarily large girth and with chromatic number at least  $(1/2 - o(1))d/\ln d$ . After a series of partial results by a number of researchers [5, 8, 27], [17, Section 4.6], Kim [24] proved an asymptotically sharp upper bound, which exceeds the lower bound only by a factor of 2:

**Theorem 1.11** (Kim [24]; see also [31, Chapter 12]). *Let  $G$  be a graph with maximum degree  $d \in \mathbb{N}$ . If  $g(G) \geq 5$ , then*

$$\chi(G) \leq (1 + o(1)) \frac{d}{\ln d}.$$

Shortly after that, Johansson [18] pushed the girth requirement further down and extended Kim’s result (modulo a constant factor) to triangle-free graphs.

**Theorem 1.12** (Johansson [18]; see also [31, Chapter 13]). *Let  $G$  be a graph with maximum degree  $d \in \mathbb{N}$ . If  $g(G) \geq 4$ , then*

$$\chi(G) = O\left(\frac{d}{\ln d}\right).$$

The proofs of Kim’s and Johansson’s theorems are examples of a particular general approach to coloring problems. The key idea is to use the LLL not once, like Erdős and Lovász did in their proof of Theorem 1.10, but several times in a sequence. On each stage, the LLL produces only a partial coloring of  $G$ —but this coloring is also made to satisfy some additional requirements. These requirements allow the process to be iterated, until finally the uncolored part of the graph becomes so sparse that a single application of the LLL can finish the proof. Dealing with such iterated applications of the LLL will be one of the main

---

<sup>8</sup>The best currently known bound that guarantees that  $\mathcal{H}$  is 2-colorable is  $d \leq c(k/\ln k)^{1/2}2^k$  for some absolute constant  $c > 0$ , due to Radhakrishnan and Srinivasan [35, Theorem 4.2]. Their proof also relies on the LLL.

difficulties we will have to overcome in Section 5. An interested reader is referred to [31] for an excellent exposition of both proofs; we also discuss them briefly in Appendix B (omitting most of the details).

**Kahn’s theorem** As we mentioned in Subsection 1.1, Vizing’s theorem asserts that if  $\Delta(G)$  is finite, then  $\chi'(G) \leq \Delta(G) + 1$ . There are several known proofs of Vizing’s theorem, none of them using the LLL.

An important generalization of graph coloring, so-called list coloring, was introduced independently by Vizing [40] and Erdős, Rubin, and Taylor [15]. Let  $G$  be a graph with vertex set  $X$ . A *list assignment* for  $G$  is a function  $L: X \rightarrow \text{Pow}(Y)$ , where  $Y$  is a set and  $\text{Pow}(Y)$  denotes its powerset. An  $L$ -*coloring* of  $G$  is a map  $f: X \rightarrow Y$  such that for all  $x \in X$ ,  $f(x) \in L(x)$ . The *list chromatic number* of  $G$  (notation:  $\chi_\ell(G)$ ) is the least  $k$  such that  $G$  admits a proper  $L$ -coloring whenever  $|L(x)| \geq k$  for all  $x \in X$ . Clearly,  $\chi_\ell(G) \geq \chi(G)$  since if  $L(x) = Y$  for all  $x \in X$ , then an  $L$ -coloring is simply a coloring with  $Y$  as the set of colors. Somewhat surprisingly, this inequality can be strict; in fact, there can be no upper bound on  $\chi_\ell(G)$  in terms of  $\chi(G)$ : there exist bipartite graphs with arbitrarily large list chromatic numbers.

*List edge colorings* and the *list chromatic index*  $\chi'_\ell(G)$  of a graph  $G$  are defined similarly, *mutatis mutandis*. The following conjecture is one of the major open problems in graph theory:

**Conjecture 1.13 (The List Edge Coloring Conjecture;** see [4, Conjecture 17.8]). *For every finite graph  $G$ ,*

$$\chi'_\ell(G) = \chi_\ell(G).$$

As a partial result toward establishing the truth of Conjecture 1.13, Kahn [19] proved the following asymptotic version of Vizing’s theorem for list colorings:

**Theorem 1.14** (Kahn [19]; see also [31, Chapter 14]). *Let  $G$  be a graph with maximum degree  $d \in \mathbb{N}$ . Then*

$$\chi'_\ell(G) = d + o(d).$$

What is important for us is that, in contrast to Vizing’s theorem, Kahn’s proof is based on the LLL; in fact, it is similar to the proofs of Kim’s and Johansson’s theorems in that it uses iterated applications of the LLL to produce partial colorings with some additional “good” properties. Note that Kahn’s theorem provides an LLL-based proof of the bound  $\chi'(G) = d + o(d)$  for *ordinary* edge colorings as well.

**Nonrepetitive and acyclic colorings** The LLL can be also applied to produce upper bounds on more “exotic” types of chromatic numbers. Here we only mention two examples. A nonempty finite sequence  $s$  is *nonrepetitive* if it cannot be decomposed as  $s = u \hat{\ } v \hat{\ } v \hat{\ } w$  for some finite sequences  $u, v, w$  with  $v \neq \emptyset$  (here  $\hat{\ }$  denotes concatenation of finite sequences). A coloring  $f$  of a graph  $G$  with vertex set  $X$  is *nonrepetitive* if for any finite path  $x_1, \dots, x_k$  in  $G$ , the sequence  $(f(x_1), \dots, f(x_k))$  is nonrepetitive. Note that a nonrepetitive coloring is, in particular, proper since if  $xGy$  and  $f(x) = f(y)$ , then the sequence  $(f(x), f(y))$ , corresponding to the path  $x, y$  of length one, contains a repetition. The least number of colors

necessary to color  $G$  nonrepetitively is called the *Thue*<sup>9</sup> number of  $G$  and is denoted by  $\pi(G)$ . The following theorem of Alon, Grytczuk, Hałuszczak, and Riordan [1] gives an upper bound on  $\pi(G)$  in terms of  $\Delta(G)$ :

**Theorem 1.15** (Alon–Grytczuk–Hałuszczak–Riordan [1, Theorem 1]). *Let  $G$  be a graph with maximum degree  $d \in \mathbb{N}$ . Then*

$$\pi(G) = O(d^2).$$

A proper coloring  $f$  of a graph  $G$  is *acyclic* if every cycle in  $G$  receives at least three different colors. The least number of colors needed for an acyclic proper coloring of  $G$  is called the *acyclic chromatic number* of  $G$  and is denoted by  $a(G)$ . In 1976, Erdős conjectured that  $a(G) = o(\Delta(G)^2)$ ; 15 years later, Alon, McDiarmid, and Reed [2] confirmed Erdős’s hypothesis.

**Theorem 1.16** (Alon–McDiarmid–Reed [2, Theorem 1.1]). *Let  $G$  be a graph with maximum degree  $d \in \mathbb{N}$ . Then*

$$a(G) = O(d^{4/3}).$$

Each one of Theorems 1.15 and 1.16 is proved by a single application of Theorem 1.8 to a carefully constructed correct instance.

### 1.3 Overview of our main results and the structure of the paper

This paper addresses the following natural question: Suppose  $(X, \mu)$  is a standard probability space and  $B$  is a correct instance over  $X$ . When can we guarantee that there is a “large” (in terms of  $\mu$ ) Borel subset of  $X$  on which  $B$  admits a Borel solution?

In our investigation, we will rely heavily on the effective proof of the variable version of the LLL due to Moser and Tardos [33]. The original motivation behind Moser and Tardos’s work was to develop an effective randomized algorithm which, given a correct instance  $B$  over a finite set  $X$ , finds a solution for  $B$ . It turns out that the Moser–Tardos algorithm naturally extends to the case when  $X$  is infinite. In Section 2 we describe (a generalized version of) the Moser–Tardos algorithm and consider its behavior in the Borel setting.

By definition, an instance over a set  $X$  is a set of constraints put on a map  $f: X \rightarrow [0; 1]$ . For example, if  $X$  is the vertex set of a graph  $G$ , then solving instances over  $X$  can help us to find vertex colorings of  $G$  with certain properties. However, sometimes we want to consider maps defined on the edges of  $G$  (i.e., edge colorings) or maybe on some other combinatorial structures constructed from  $G$  (such as paths of length 2, or cycles, etc.). Additionally, even when looking for vertex colorings, we sometimes want to assign to each vertex several colors at once, which can be viewed as replacing each element of  $X$  by countably many “copies” of it and coloring each “copy” independently. In order to cover all potential combinatorial applications, we enlarge the set  $X$ , adding points for various combinatorial data that can be built from the elements of  $X$ . We call the resulting “universal” combinatorial structure the *amplification* of  $X$  and denote it by  $\text{HF}_0(X)$  (here the letters “HF” stand for “hereditarily

---

<sup>9</sup>Thue initiated the study of nonrepetitive sequences. While it is easy to see that there are no nonrepetitive sequences of length 4 over an alphabet of size 2, Thue’s famous theorem [37] asserts that there exist arbitrarily long nonrepetitive sequences over an alphabet of size 3.

finite”). The construction of  $\text{HF}_0(X)$  is described in Section 3. All our results are stated for instances over  $\text{HF}_0(X)$ ; however, to simplify the current discussion, we will only be talking about instances over  $X$  in this subsection.

Our first main result is the approximate Borel LLL, which we state and prove in Section 4. Let  $(X, \mu)$  be a standard probability space. Suppose that  $B$  is a correct Borel instance over  $X$ . We can measure how well a function  $f: X \rightarrow [0; 1]$  solves  $B$  by looking at its *defect*, i.e., the set

$$D(f, B) := \{x \in X : \exists S \in \text{dom}_x(B)(f|_S \in B_S)\}.$$

Theorem 4.1, the approximate Borel LLL, asserts that for every  $\varepsilon > 0$ , there is a Borel map  $f: X \rightarrow [0; 1]$  such that  $\mu(D(f, B)) < \varepsilon$ , provided that  $B$  is *locally finite*, i.e., for all  $x \in X$ , the set  $\text{dom}_x(B)$  is finite. Most (but not all) standard applications of the LLL only consider locally finite instances. Of the examples listed in Subsection 1.2, Theorems 1.10, 1.11, 1.12, and 1.14 only use locally finite instances; in particular, Theorem 4.1 immediately yields Theorems 1.4 and 1.5 on approximate chromatic numbers of Borel graphs. On the other hand, Theorems 1.15 and 1.16 apply the LLL to non-locally finite instances: there can be infinitely many paths or cycles passing through a given vertex in a locally finite graph. We would like to point out that in their recent work [12], which was carried out independently of this one, Csóka, Grabowski, Máthé, Pikhurko, and Tyros use an approach similar to ours in order to establish a *Borel* version of the LLL for the class of instances satisfying stronger finiteness assumptions (namely having uniformly subexponential growth).

Our second main result is the measurable version of the LLL for probability measure-preserving actions of countable groups, which we present in Section 5. It shows that under certain additional restrictions on the instance  $B$ , we can find a Borel solution that works on a *conull* subset—even if  $B$  is not locally finite. To motivate these restrictions, consider a graph  $G$  with vertex set  $X$ . Combinatorial problems related to  $G$  usually require solving instances of the LLL that possess the following two properties:

1. the correctness of a solution can be verified separately within each connected component of  $G$ ; for example,  $f: X \rightarrow Y$  is a proper coloring of  $G$  if and only if  $f|_C$  is a proper coloring of  $G|_C$  for each connected component  $C$  of  $G$ ;
2. the instance only depends on the graph structure of  $G$ ; in other words, it is invariant under the (combinatorial) automorphisms of  $G$ .

To capture these two properties, we will introduce the notion of an *isomorphism structure*  $\mathcal{I}$  on an equivalence relation  $E$  on a set  $X$ . Roughly speaking, an isomorphism structure on  $E$  is a collection of bijections of the form  $\varphi: X_1 \rightarrow X_2$ , where  $X_1$  and  $X_2$  are contained entirely within  $E$ -classes, which can be thought of as isomorphisms between certain substructures of  $X$ . For example, if  $G$  is a graph with vertex set  $X$  and  $E_G$  is the equivalence relation whose classes are the connected components of  $G$ , then the set  $\mathcal{I}_G$  of all isomorphisms between connected components of  $G$  is an isomorphism structure on  $E_G$ . Similarly, if  $a: \Gamma \curvearrowright X$  is an action of a countable group  $\Gamma$  on  $X$  and  $E_a$  is the corresponding orbit equivalence relation, then the set of all  $\Gamma$ -equivariant bijections between orbits forms an isomorphism structure on  $E_a$ , which we denote by  $\mathcal{I}_a$ .

Now, let  $a: \Gamma \curvearrowright (X, \mu)$  be a measure-preserving action of a countable group  $\Gamma$  on a standard probability space  $(X, \mu)$  that admits a factor map to the  $[0; 1]$ -shift action of  $\Gamma$ .

Consider a correct Borel instance  $B$  over  $X$  such that (a) if  $S \in \text{dom}(B)$ , then  $S$  is contained entirely within a single orbit; and (b)  $B$  is invariant under  $\mathcal{I}_a$ . A part of Theorem 5.4 (Lemma 5.17, to be precise) asserts that such  $B$  admits a Borel solution on an invariant conull Borel subset of  $X$ . This is enough to conclude that graphs of the form  $G(a, S)$ , where  $S \subseteq \Gamma$  is a generating set, satisfy measurable analogs of Theorems 1.15 and 1.16. However, to prove Theorems 1.1 and 1.3, a stronger measurable version of the LLL is needed: as we discussed in Subsection 1.2, the proofs of their combinatorial counterparts (namely Theorems 1.11, 1.12, and 1.14) require solving several instances of the LLL in a sequence, where each next instance is built *using the solution of the previous one*. In other words, even though the very first instance  $B_0$  is  $\mathcal{I}_a$ -invariant, as long as we find a solution  $f$  for it, the next instance  $B_1$  will be invariant only under those maps from  $\mathcal{I}_a$  that preserve the value of  $f$ .

To formalize this complication, we define the *LLL game* between two players. On his first turn, Player I chooses an  $\mathcal{I}_a$ -invariant correct Borel instance  $B_0$ . Player II responds (if he can) by choosing a Borel (partial, i.e., defined on a  $\Gamma$ -invariant conull subset) solution  $f_0$  for  $B_0$ . Next, Player I chooses a new correct Borel instance  $B_1$ , this time only invariant under those maps from  $\mathcal{I}_a$  that preserve  $f_0$ . Player II has to respond by finding a Borel (partial) solution  $f_1$  for  $B_1$ . On Step 2, Player I chooses a correct instance  $B_2$  invariant under the bijections from  $\mathcal{I}_a$  that preserve both  $f_0$  and  $f_1$ , etc. The actual statement of Theorem 5.4 is that Player II has a winning strategy in this game.

Finally, we turn to the following question: Is it necessary to assume that  $a$  admits a factor map to the  $[0; 1]$ -shift action in order to establish Theorem 5.4, or is this assumption just an artifact of our proof? In Section 6 (Theorems 6.1 and 6.1') we show that this assumption is, in fact, necessary, at least for actions of amenable groups. Indeed, we prove that a probability measure-preserving free ergodic action  $a: \Gamma \curvearrowright (X, \mu)$  of a countably infinite amenable group  $\Gamma$  satisfies the measurable version of the LLL *if and only if* it admits a factor map to the shift action  $\Gamma \curvearrowright ([0; 1]^\Gamma, \lambda^\Gamma)$ . Moreover, we show that for actions that do not admit such a factor map, even a much weaker version of the LLL than the one in Theorem 5.4 fails. Theorems 6.1 and 6.1' also show that the local finiteness assumption in Theorem 4.1 is necessary.

## 1.4 General notation, conventions, and preliminary results

We use  $\mathbb{N}$  to denote the set of all nonnegative integers.

Our standard references for descriptive set theory are [20] and [38]. Below we only list the most basic facts and terminology used throughout the paper without mention.

A *standard Borel space*  $(X, \mathfrak{B})$  is a set  $X$  together with a  $\sigma$ -algebra  $\mathfrak{B}$  of *Borel sets* such that there is a compatible Polish (i.e., separable completely metrizable) topology  $\tau$  on  $X$  with  $\mathfrak{B}$  as its  $\sigma$ -algebra of Borel sets. We will suppress the notation for the  $\sigma$ -algebra and denote a standard Borel space  $(X, \mathfrak{B})$  simply by  $X$ . A function  $f: X \rightarrow Y$  between standard Borel spaces  $X$  and  $Y$  is *Borel* if  $f$ -preimages of Borel subsets of  $Y$  are Borel in  $X$ . Due to the Borel isomorphism theorem [38, Theorem 13.10], all countable standard Borel spaces are discrete and all uncountable ones are isomorphic to each other.

We use  $P(X)$  to denote the set of all probability Borel measures on a standard Borel space  $X$ . If  $\mu \in P(X)$ , then the pair  $(X, \mu)$  is called a *standard probability space*. A measure  $\mu \in P(X)$  is *atomless* if  $\mu(\{x\}) = 0$  for all  $x \in X$ . The measure isomorphism theorem [38,

Theorem 10.6] asserts that all standard probability spaces  $(X, \mu)$  with atomless  $\mu$  are Borel isomorphic. If  $X$  is a standard Borel space and  $X' \subseteq X$  is a Borel set, then we view  $P(X')$  as a subset of  $P(X)$ . In particular, if  $\mu \in P(X')$ , then we also use  $\mu$  to denote the extension of  $\mu$  to  $X$  (i.e., the pushforward  $\iota_*(\mu)$  of  $\mu$  under the inclusion map  $\iota: X' \hookrightarrow X$ ); similarly, if  $\mu \in P(X)$  and  $X'$  is  $\mu$ -conull, then we use  $\mu$  to denote the restriction of  $\mu$  to  $X'$ . The Lebesgue measure on the unit interval  $[0; 1]$  is denoted by  $\lambda$ .

A subset  $A$  of a standard Borel space  $X$  is *analytic* if it is an image of a Borel set under a Borel function. Informally, a set is analytic if it can be defined using existential (but not universal) quantifiers. Analytic subsets of  $X$  are universally measurable, i.e.,  $\mu$ -measurable for every  $\mu \in P(X)$  [38, Corollary 14.10]. The complement of an analytic set is said to be *co-analytic*. If a set is both analytic and co-analytic, then it is Borel [38, Corollary 12.7].

Recall that for a set  $X$ ,  $[X]^{<\infty}$  denotes the set of all nonempty finite subsets of  $X$ . If  $X$  is a standard Borel space, then  $[X]^{<\infty}$  is also naturally equipped with a standard Borel structure. One way to see this is to notice that if  $\tau$  is a compatible Polish topology on  $X$ , then  $[X]^{<\infty}$  is a Borel subset of  $\mathcal{K}(X, \tau)$ , the Polish space of all compact subsets of  $(X, \tau)$  equipped with the Vietoris topology [38, Subsection 3.D]. For a standard Borel space  $X$ , there is a Borel map  $f: [X]^{<\infty} \rightarrow X$  such that for all  $S \in [X]^{<\infty}$ ,  $f(S) \in S$ ; for example, if  $<$  is a Borel linear order on  $X$  (which exists due to the Borel isomorphism theorem), then the function  $f: S \mapsto \min S$  is Borel.

For sets  $X, Y$ , elements  $x \in X, y \in Y$ , and a subset  $A \subseteq X \times Y$ , we use the following notation:

$$\begin{aligned} A_x &:= \{y \in Y : (x, y) \in A\}; \\ A^y &:= \{x \in X : (x, y) \in A\}. \end{aligned}$$

The following fundamental result is used without mention:

**Theorem 1.17 (The Luzin–Novikov theorem;** see [20, Theorem 18.10]). *Let  $X, Y$  be standard Borel spaces and let  $A \subseteq X \times Y$  be a Borel set such that for all  $x \in X$ , the set  $A_x$  is countable. Then  $A = \bigcup_{n=0}^{\infty} A_n$ , where the sets  $(A_n)_{n=0}^{\infty}$  are disjoint and for each  $n \in \mathbb{N}$  and  $x \in X$ ,  $|(A_n)_x| \leq 1$ . In particular, the set  $\text{proj}_X(A) := \{x \in X : A_x \neq \emptyset\}$  is Borel.*

Informally, the Luzin–Novikov theorem implies that if a set is defined only using quantifiers ranging over countable sets, then it is Borel.

On a couple of occasions, we will need the following fact.

**Proposition 1.18** (Countable colorings of locally finite graphs). *Let  $G$  be a locally finite analytic graph on a standard Borel space  $X$ . Then  $\chi_B(G) \leq \aleph_0$ .*

*Proof.* Let  $(B_n)_{n=0}^{\infty}$  be a countable family of Borel subsets of  $X$  that separates points and is closed under complements and finite intersections. In particular, for any  $x \in X$  and  $S \subseteq X \setminus \{x\}$ , if  $S$  is finite, then there is  $n \in \mathbb{N}$  such that  $x \in B_n$  but  $S \cap B_n = \emptyset$ .

Define a set  $Z \subseteq X \times \mathbb{N}$  as follows:

$$\begin{aligned} (x, n) \in Z &\iff x \in B_n \text{ and } G_x \cap B_n = \emptyset \\ &\iff x \in B_n \text{ and } \forall y \in B_n (y \notin G_x). \end{aligned}$$

The second line in the above definition makes it clear that the set  $Z$  is co-analytic. Since for all  $x \in X$ , there is  $n \in \mathbb{N}$  such that  $(x, n) \in Z$ , due to the Novikov separation theorem (see [20, Theorem 28.5]), there is a Borel function  $f: X \rightarrow \mathbb{N}$  such that for all  $x \in X$ ,  $(x, f(x)) \in Z$ . It is easy to see that  $f$  is a Borel proper coloring of  $G$ , as desired. ■

One can also deduce Proposition 1.18 from the general characterization of analytic graphs with countable Borel chromatic numbers due to Kechris, Solecki, and Todorcevic [23, Theorem 6.3].

## 2 Moser–Tardos theory

As we mentioned in the introduction, a major role in our arguments will be played by the ideas coming from the effective proof of the variable version of the LLL due to Moser and Tardos [33]. In this section we review their method and introduce some convenient notation and terminology. Most of the results in this section (apart from Corollary 2.6) are essentially present in [33] (although Moser and Tardos state them in a different language). Note that Moser and Tardos, motivated by algorithmic applications, only consider the case when the ground set  $X$  is finite; however, as Kun noticed in [26], most of their results generalize to the case of arbitrary  $X$ .

First we discuss informally the idea of Moser and Tardos’s approach. Suppose for a moment that  $X$  is a finite set and  $B$  is a correct instance over  $X$ . We want to find a solution for  $B$ , i.e., a function  $f: X \rightarrow [0; 1]$  with  $f|S \notin B_S$  for all  $S \in \text{dom}(B)$ . Let  $(\vartheta_n)_{n=0}^\infty$  be a sequence of functions  $\vartheta_n: X \rightarrow [0; 1]$ , where each  $\vartheta_n$  is chosen randomly and independently from  $[0; 1]^X$  (according to the measure  $\lambda^X$ ). Set  $f_0 := \vartheta_0$ . If  $f_0$  is itself a solution for  $B$ , then we are done, so assume that the set

$$A'_0 := \{S \in \text{dom}(B) : f_0|S \in B_S\}$$

is nonempty. Now we use  $\vartheta_1$  to adjust  $f_0$  by changing some (but not necessarily all) of the values it assigns to the elements of  $X$ . Let  $A_0$  be any maximal disjoint subset of  $A'_0$  and define

$$X_0 := \{x \in X : \exists S \in A_0 (x \in S)\}.$$

It is easy to see that for all  $S \in [X \setminus X_0]^{<\infty}$ ,  $f_0|S \notin B_S$ . This observation motivates keeping the values assigned to the elements of  $X \setminus X_0$  unchanged, so we set

$$f_1(x) := \begin{cases} \vartheta_1(x) & \text{if } x \in X_0; \\ f_0(x) & \text{otherwise.} \end{cases}$$

Continuing this process, we obtain a (finite or infinite) sequence  $f_0, f_1, \dots$  of maps from  $X$  to  $[0; 1]$ . Moser and Tardos proved that if  $X$  is finite, then this sequence is finite with probability 1, meaning that for almost every choice of  $(\vartheta_n)_{n=0}^\infty$ , the process eventually terminates and outputs a solution for  $B$ .

Let us now proceed to formal definitions. For the rest of this section, we fix a set  $X$  and a correct instance  $B$  over  $X$ .

**Definition 2.1** (Moser–Tardos processes). A *table* is a map  $\vartheta: X \rightarrow [0; 1]^{\mathbb{N}}$ . For  $x \in X$  and  $n \in \mathbb{N}$ , we write  $\vartheta(x, n)$  for  $\vartheta(x)(n)$ . Let  $\mathcal{A} = (A_n)_{n=0}^{\infty}$  be a sequence of subsets of  $\text{dom}(B)$ . For all  $x \in X$ , let  $t_0(x) := 0$ ; and for all  $n \in \mathbb{N}$  and  $x \in X$ , let

$$t_{n+1}(x) := \begin{cases} t_n(x) + 1 & \text{if there is } S \in A_n \text{ with } x \in S; \\ t_n(x) & \text{otherwise.} \end{cases}$$

For all  $x \in X$  and  $n \in \mathbb{N}$ , let  $f_n(x) := \vartheta(x, t_n(x))$ . For all  $n \in \mathbb{N}$ , let

$$A'_n := \{S \in \text{dom}(B) : f_n|_S \in B_S\}.$$

The sequence  $\mathcal{A}$  is a *Moser–Tardos process with input  $\vartheta$*  if for each  $n \in \mathbb{N}$ ,  $A_n$  is a maximal disjoint subset of  $A'_n$ .

It is immediate from the above definition that for every table  $\vartheta: X \rightarrow [0; 1]^{\mathbb{N}}$ , there exists a Moser–Tardos process with input  $\vartheta$  (it can be constructed inductively by choosing each  $A_n$  to be a maximal disjoint subset of  $A'_n$ ). The disjointness condition is used, in particular, to ensure that for every  $x \in X$  with  $t_{n+1}(x) > t_n(x)$ , there is a *unique*  $S \in A_n$  such that  $x \in S$ .

**Proposition 2.2.** *Let  $\mathcal{A} = (A_n)_{n=0}^{\infty}$  be a Moser–Tardos process. For  $n \in \mathbb{N}$ , let*

$$X_n := \{x \in X : \exists S \in A_n(x \in S)\}.$$

*Then for all  $S \in [X \setminus X_n]^{<\infty}$ ,  $f_n|_S \notin B_S$ .*

*Proof.* Let  $S \in [X \setminus X_n]^{<\infty}$ . By the definition of  $X_n$ ,  $S$  is disjoint from all  $S' \in A_n$ . Since  $A_n$  is a maximal disjoint subset of  $A'_n$ ,  $S \notin A'_n$ , i.e.,  $f_n|_S \notin B_S$ , as desired. ■

Suppose that  $\mathcal{A} = (A_n)_{n=0}^{\infty}$  is a Moser–Tardos process with input  $\vartheta: X \rightarrow [0; 1]^{\mathbb{N}}$ . By definition, the sequence  $(t_n(x))_{n=0}^{\infty}$  is monotone increasing for all  $x \in X$ . We say that an element  $x \in X$  is  *$\mathcal{A}$ -stable* if the sequence  $(t_n(x))_{n=0}^{\infty}$  is eventually constant. Let  $\text{Stab}(\mathcal{A}) \subseteq X$  be the set of all  $\mathcal{A}$ -stable elements of  $X$ . Define  $t: \text{Stab}(\mathcal{A}) \rightarrow \mathbb{N}$  by

$$t(x) := \lim_{n \rightarrow \infty} t_n(x),$$

and define  $f: \text{Stab}(\mathcal{A}) \rightarrow [0; 1]$  by

$$f(x) := \vartheta(x, t(x)).$$

Then we have the following limit analog of Proposition 2.2:

**Proposition 2.3.** *If  $\mathcal{A} = (A_n)_{n=0}^{\infty}$  is a Moser–Tardos process, then for any nonempty finite set  $S$  of  $\mathcal{A}$ -stable elements,  $f|_S \notin B_S$ .*

*Proof.* Let  $S \in [\text{Stab}(\mathcal{A})]^{<\infty}$  and let  $m \in \mathbb{N}$  be such that for all  $n \geq m$  and for all  $x \in S$ ,  $t_n(x) = t_m(x)$ . Then  $f|_S = f_m|_S$ . Note that  $S$  is disjoint from all  $S' \in A_m$ . Indeed, if  $x \in S' \cap S$  for  $S' \in A_m$ , then  $t_{m+1}(x) = t_m(x) + 1$ ; a contradiction with the choice of  $m$ . Since  $A_m$  is a maximal disjoint subset of  $A'_m$ ,  $S \notin A'_m$ , i.e.,  $f_m|_S \notin B_S$ , as desired. ■



For each  $S \in \text{dom}(B)$ , let the quantity  $N(S, \mathcal{A}) \in \mathbb{N} \cup \{\infty\}$  be given by

$$N(S, \mathcal{A}) := |\{n \in \mathbb{N} : S \in A_n\}|.$$

Note that for all  $x \in X$ ,

$$\lim_{n \rightarrow \infty} t_n(x) = \sum_{S \in \text{dom}_x(B)} N(S, \mathcal{A}), \quad (2.1)$$

so  $x$  is  $\mathcal{A}$ -stable if and only if the sum on the right-hand side of (2.1) is finite. Our goal therefore is to obtain sufficiently good upper bounds on the numbers  $N(S, \mathcal{A})$ . For this we look at certain patterns in  $\vartheta$ .

A *pile* is a nonempty finite set  $\mathcal{P}$  of functions of the form  $\tau: S \rightarrow \mathbb{N}$  with  $S \in \text{dom}(B)$ , satisfying the following conditions:

1. the graphs of the elements of  $\mathcal{P}$  are disjoint; in other words, for every  $\tau, \tau' \in \mathcal{P}$  and  $x \in \text{dom}(\tau) \cap \text{dom}(\tau')$ , either  $\tau = \tau'$ , or else,  $\tau(x) \neq \tau'(x)$ ;
2. for each  $\tau \in \mathcal{P}$  and  $x \in \text{dom}(\tau)$ , either  $\tau(x) = 0$ , or else, there exists  $\tau' \in \mathcal{P}$  with  $x \in \text{dom}(\tau')$  and  $\tau'(x) = \tau(x) - 1$ .

The *support* of a pile  $\mathcal{P}$  is the set

$$\text{supp}(\mathcal{P}) := \bigcup_{\tau \in \mathcal{P}} \text{dom}(\tau).$$

Note that  $\text{supp}(\mathcal{P}) \in [X]^{<\infty}$ .

For a pile  $\mathcal{P}$  and  $\tau, \tau' \in \mathcal{P}$ , we say that  $\tau'$  *supports*  $\tau$  (notation:  $\tau' \prec \tau$ ) if there is  $x \in \text{dom}(\tau) \cap \text{dom}(\tau')$  with  $\tau'(x) = \tau(x) - 1$ . A pile  $\mathcal{P}$  is *neat* if there is no sequence  $\tau_1, \tau_2, \dots, \tau_k \in \mathcal{P}$  with  $2 \leq k \in \mathbb{N}$  such that  $\tau_1 \prec \tau_2 \prec \dots \prec \tau_k \prec \tau_1$ . Equivalently,  $\mathcal{P}$  is neat if the transitive closure of the relation  $\prec$  on  $\mathcal{P}$  is a (strict) partial order. A *top element* in a pile  $\mathcal{P}$  is any  $\tau \in \mathcal{P}$  for which there is no  $\tau' \in \mathcal{P}$  such that  $\tau \prec \tau'$ . The set of all top elements in  $\mathcal{P}$  is denoted by  $\text{Top}(\mathcal{P})$ . Note that if  $\mathcal{P}$  is a neat pile, then  $\text{Top}(\mathcal{P}) \neq \emptyset$ . The *height*  $h(\mathcal{P})$  of a neat pile  $\mathcal{P}$  is the largest  $k \in \mathbb{N}$  such that there is a sequence  $\tau_1, \dots, \tau_k \in \mathcal{P}$  with  $\tau_1 \prec \dots \prec \tau_k$  (so necessarily  $h(\mathcal{P}) \geq 1$ ).

For a table  $\vartheta: X \rightarrow [0; 1]^{\mathbb{N}}$  and a function  $\tau: S \rightarrow \mathbb{N}$ , where  $S \in [X]^{<\infty}$ , the  $\vartheta$ -*value* of  $\tau$  is the map  $\vartheta(\tau): S \rightarrow [0; 1]$  defined by

$$\vartheta(\tau)(x) := \vartheta(x, \tau(x)).$$

We say that a pile  $\mathcal{P}$  *appears* in a table  $\vartheta$  if for all  $\tau \in \mathcal{P}$ ,  $\vartheta(\tau) \in B_{\text{dom}(\tau)}$ .

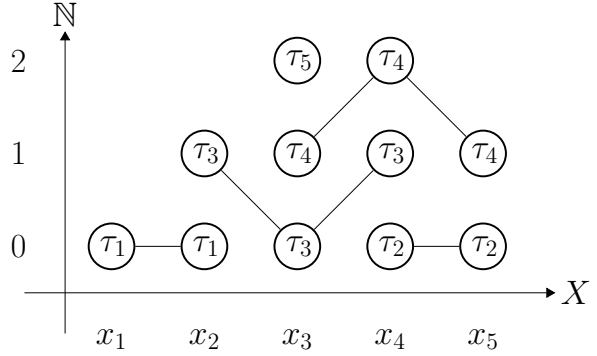


Figure 1:  $\mathcal{P} = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$  is a neat pile of height 4 with  $\text{supp}(\mathcal{P}) = \{x_1, x_2, x_3, x_4, x_5\}$  and  $\text{Top}(\mathcal{P}) = \{\tau_5\}$ .

For  $S \in \text{dom}(B)$ , let  $\mathfrak{P}(S)$  denote the set of all neat piles  $\mathcal{P}$  with  $\text{Top}(\mathcal{P}) = \{\tau\}$ , where the unique top element  $\tau$  of  $\mathcal{P}$  satisfies  $\text{dom}(\tau) = S$ . Note that since  $B$  is locally countable,  $\mathfrak{P}(S)$  is countable for all  $S \in \text{dom}(B)$ . Define  $M(S, \vartheta) \in \mathbb{N} \cup \{\infty\}$  by

$$M(S, \vartheta) := |\{\mathcal{P} \in \mathfrak{P}(S) : \mathcal{P} \text{ appears in } \vartheta\}|.$$

The next proposition gives an upper bound on  $N(S, \mathcal{A})$ , where  $S \in \text{dom}(B)$  and  $\mathcal{A}$  is a Moser–Tardos process with input  $\vartheta$ , in terms of  $S$  and  $\vartheta$  alone (i.e., independent of  $\mathcal{A}$ ).

**Proposition 2.4.** *Let  $\mathcal{A} = (A_n)_{n=0}^\infty$  be a Moser–Tardos process with input  $\vartheta$  and let  $S \in \text{dom}(B)$ . Then for all  $n \in \mathbb{N}$  such that  $S \in A'_n$ , there is a neat pile  $\mathcal{P} \in \mathfrak{P}(S)$  of height exactly  $n + 1$  such that  $\mathcal{P}$  appears in  $\vartheta$ . In particular,  $N(S, \mathcal{A}) \leq M(S, \vartheta)$ .*

*Proof.* Suppose  $S \in A'_n$ . We construct  $\mathcal{P}$  as follows. Let  $\mathcal{P}^{(n)} := \{t_n|S\}$ . For each  $n > k \geq 0$ , define  $\mathcal{P}^{(k)}$  inductively via

$$\mathcal{P}^{(k)} := \left\{ t_k|T : T \in A_k \text{ and } \exists \tau \in \bigcup_{i=k+1}^n \mathcal{P}^{(i)} (\text{dom}(\tau) \cap T \neq \emptyset) \right\}.$$

Let  $\mathcal{P} := \bigcup_{k=0}^n \mathcal{P}^{(k)}$ .

**Claim 2.4.1.** *The graphs of the elements of  $\mathcal{P}$  are disjoint.*

*Proof.* Suppose  $0 \leq k_1 \leq k_2 \leq n$ ,  $\tau_1 \in \mathcal{P}^{(k_1)}$ ,  $\tau_2 \in \mathcal{P}^{(k_2)}$ ,  $\tau_1 \neq \tau_2$ , and for some  $x \in \text{dom}(\tau_1) \cap \text{dom}(\tau_2)$ ,  $\tau_1(x) = \tau_2(x)$ . If  $k_1 = k_2 = k$ , then  $k < n$  (for  $|\mathcal{P}^{(n)}| = 1$ ), so  $\text{dom}(\tau_1)$  and  $\text{dom}(\tau_2)$  are distinct, hence disjoint, elements of  $A_k$ . Therefore,  $k_1 < k_2$ . Since  $\text{dom}(\tau_1) \in A_{k_1}$ , we get  $\tau_2(x) = t_{k_2}(x) \geq t_{k_1+1}(x) = t_{k_1}(x) + 1 = \tau_1(x) + 1$ ; a contradiction.  $\dashv$

**Claim 2.4.2.**  *$\mathcal{P}$  is a pile.*

*Proof.* Thanks to Claim 2.4.1, we only need to check that if  $\tau \in \mathcal{P}$ ,  $x \in \text{dom}(\tau)$ , and  $\tau(x) > 0$ , then there is  $\tau' \in \mathcal{P}$  with  $x \in \text{dom}(\tau')$  and  $\tau'(x) = \tau(x) - 1$ . Indeed, suppose that  $\tau \in \mathcal{P}^{(k)}$  for some  $0 \leq k \leq n$ . This means that  $t_k(x) > 0$ . Since  $t_{i+1}(x) \leq t_i(x) + 1$  for all  $i \in \mathbb{N}$ , there is  $\ell < k$  such that  $t_\ell(x) = t_k(x) - 1$  and  $t_{\ell+1}(x) = t_k(x)$ . Therefore, there is  $T \in A_\ell$  with  $x \in T$ . In particular,  $\text{dom}(\tau) \cap T \neq \emptyset$ , so  $t_\ell|T \in \mathcal{P}^{(\ell)}$ , and we are done.  $\dashv$

**Claim 2.4.3.** *Let  $\tau_1 \in \mathcal{P}^{(k_1)}$ ,  $\tau_2 \in \mathcal{P}^{(k_2)}$ . If  $x \in \text{dom}(\tau_1) \cap \text{dom}(\tau_2)$ , then  $\tau_1(x) < \tau_2(x) \iff k_1 < k_2$ . In particular, if  $\tau_1 \prec \tau_2$ , then  $k_1 < k_2$ ; hence  $\mathcal{P}$  is neat.*

*Proof.* This claim follows immediately from the fact that for each  $x \in X$ , the sequence  $(t_k(x))_{k=0}^\infty$  is monotone increasing.  $\dashv$

**Claim 2.4.4.**  *$\text{Top}(\mathcal{P}) = \{t_n|S\}$ . In particular,  $\mathcal{P} \in \mathfrak{P}(S)$ .*

*Proof.* Due to claim 2.4.3,  $t_n|S$  is a top element in  $\mathcal{P}$ . If  $\tau \in \mathcal{P}^{(k)}$  for some  $0 \leq k < n$ , then for some  $\tau' \in \bigcup_{i=k+1}^n \mathcal{P}^{(i)}$ ,  $\text{dom}(\tau') \cap \text{dom}(\tau) \neq \emptyset$ . By Claim 2.4.3 again,  $\tau(x) < \tau'(x)$  for all  $x \in \text{dom}(\tau') \cap \text{dom}(\tau)$ . Therefore,  $\tau \notin \text{Top}(\mathcal{P})$ , as desired.  $\dashv$

**Claim 2.4.5.**  *$\mathcal{P}$  appears in  $\vartheta$ .*

*Proof.* Indeed, by definition, for all  $\tau \in \mathcal{P}^{(k)}$ ,  $\text{dom}(\tau) \in A'_k$ , so we have  $\vartheta(\tau) = f_k|_{\text{dom}(\tau)} \in B_{\text{dom}(\tau)}$ , as desired.  $\dashv$

**Claim 2.4.6.**  $h(\mathcal{P}) = n + 1$ .

*Proof.* Claim 2.4.3 implies that  $h(\mathcal{P}) \leq n + 1$ . To prove that  $h(\mathcal{P}) \geq n + 1$ , it is enough to show that for every  $0 \leq k < n$  and  $\tau \in \mathcal{P}^{(k+1)}$ , there is  $\tau' \in \mathcal{P}^{(k)}$  with  $\tau' \prec \tau$ . Let

$$X_k := \{x \in X : \exists T \in A_k(x \in T)\}.$$

We need to show that  $\text{dom}(\tau) \cap X_k \neq \emptyset$ . Indeed, if  $\text{dom}(\tau) \cap X_k = \emptyset$ , then  $t_{k+1}|_{\text{dom}(\tau)} = t_k|_{\text{dom}(\tau)}$ . By Proposition 2.2, we get  $f_{k+1}|_{\text{dom}(\tau)} = f_k|_{\text{dom}(\tau)} \notin B_{\text{dom}(\tau)}$ . But  $\text{dom}(\tau) \in A'_{k+1}$ ; a contradiction.  $\dashv$

Together, Claims 2.4.1–2.4.6 establish all the required properties of  $\mathcal{P}$ .  $\blacksquare$

We say that an element  $x \in X$  is  $\vartheta$ -stable if

$$\sum_{S \in \text{dom}_x(B)} M(S, \vartheta) < \infty.$$

Due to Proposition 2.4, if  $x$  is  $\vartheta$ -stable, then it is  $\mathcal{A}$ -stable for any Moser–Tardos process  $\mathcal{A}$  with input  $\vartheta$ .

Now we are going to switch the order of summation and, instead of counting how many piles from  $\mathfrak{P}(S)$  appear in a given table  $\vartheta$ , we will fix a pile  $\mathcal{P}$  and estimate the probability that it appears in a table  $\vartheta$  chosen at random. Note that for a given pile  $\mathcal{P}$ , the restriction  $\vartheta|_{\text{supp}(\mathcal{P})}$  fully determines whether  $\mathcal{P}$  appears in  $\vartheta$  or not, so let  $W(\mathcal{P}) \subseteq [0; 1]^{\text{supp}(\mathcal{P}) \times \mathbb{N}}$  denote the set such that

$$\mathcal{P} \text{ appears in } \vartheta \iff \vartheta|_{\text{supp}(\mathcal{P})} \in W(\mathcal{P}).$$

It is easy to see that the set  $W(\mathcal{P})$  is Borel. Since the graphs of the elements of  $\mathcal{P}$  are disjoint, there is a simple expression for the measure of  $W(\mathcal{P})$ ; namely, we have

$$\lambda^{\text{supp}(\mathcal{P}) \times \mathbb{N}}(W(\mathcal{P})) = \prod_{\tau \in \mathcal{P}} \lambda^{\text{dom}(\tau)}(B_{\text{dom}(\tau)}).$$

The following is the fundamental result of Moser–Tardos theory.

**Theorem 2.5.** *Let  $S \in \text{dom}(B)$ . Suppose that  $p: \text{dom}(B) \rightarrow [0; 1]$  is a function witnessing the correctness of  $B$ . Then*

$$\sum_{\mathcal{P} \in \mathfrak{P}(S)} \lambda^{\text{supp}(\mathcal{P}) \times \mathbb{N}}(W(\mathcal{P})) \leq \frac{p(S)}{1 - p(S)}. \quad (2.2)$$

We defer the proof of Theorem 2.5 until Appendix A. Now we have the following corollary:

**Corollary 2.6.** *For all  $x \in X$ , we have*

$$\sum_{S \in \text{dom}_x(B)} \sum_{\mathcal{P} \in \mathfrak{P}(S)} \lambda^{\text{supp}(\mathcal{P}) \times \mathbb{N}}(W(\mathcal{P})) < \infty.$$

*Proof.* Let  $p: \text{dom}(B) \rightarrow [0; 1]$  witness the correctness of  $B$ . Due to Theorem 2.5, it suffices to check that the sum

$$\sum_{S \in \text{dom}_x(B)} \frac{p(S)}{1 - p(S)} \quad (2.3)$$

is finite. We may assume that  $p(S) = 0$  whenever  $\lambda^S(B_S) = 0$ . If for all  $S \in \text{dom}_x(B)$ ,  $\lambda^S(B_S) = 0$ , then the sum (2.3) is 0 (hence finite). Otherwise, for some  $S_0 \in \text{dom}_x(B)$ ,  $\lambda^{S_0}(B_{S_0}) > 0$ , and thus the correctness of  $B$  implies

$$\prod_{\substack{S \in \text{dom}(B): \\ S \cap S_0 \neq \emptyset}} (1 - p(S)) > 0.$$

Therefore,

$$\sum_{S \in \text{dom}_x(B)} p(S) \leq \sum_{\substack{S \in \text{dom}(B): \\ S \cap S_0 \neq \emptyset}} p(S) < \infty.$$

In particular, for all but finitely many  $S \in \text{dom}_x(B)$ ,  $p(S) \leq 1/2$ , so

$$\frac{p(S)}{1 - p(S)} \leq 2p(S).$$

Hence, the sum (2.3) is finite, as desired. ■

The next corollary considers the case when a table  $\vartheta$  is chosen randomly from  $[0; 1]^{X \times \mathbb{N}}$ . Note that the product probability space  $([0; 1]^{X \times \mathbb{N}}, \lambda^{X \times \mathbb{N}})$  is standard only if  $X$  is countable.

**Corollary 2.7.** *For all  $x \in X$ , we have*

$$\int_{[0; 1]^{X \times \mathbb{N}}} \sum_{S \in \text{dom}_x(B)} M(S, \vartheta) d\lambda^{X \times \mathbb{N}}(\vartheta) < \infty.$$

*In particular,*

$$\lambda^{X \times \mathbb{N}}(\{\vartheta \in [0; 1]^{X \times \mathbb{N}} : x \text{ is } \vartheta\text{-stable}\}) = 1.$$

*Proof.* Due to Corollary 2.6, for all  $x \in X$ , we have

$$\begin{aligned} & \int_{[0; 1]^{X \times \mathbb{N}}} \sum_{S \in \text{dom}_x(B)} M(S, \vartheta) d\lambda^{X \times \mathbb{N}}(\vartheta) \\ &= \sum_{S \in \text{dom}_x(B)} \int_{[0; 1]^{X \times \mathbb{N}}} M(S, \vartheta) d\lambda^{X \times \mathbb{N}}(\vartheta) \\ &= \sum_{S \in \text{dom}_x(B)} \sum_{\mathcal{P} \in \mathfrak{P}(S)} \lambda^{\text{supp}(\mathcal{P}) \times \mathbb{N}}(W(\mathcal{P})) < \infty, \end{aligned}$$

as desired. ■

Let us show how Theorem 2.5 (in the form of Corollary 2.7) implies the variable version of the LLL (i.e., Theorem 1.8). Since  $B$  is locally countable, we may without loss of generality assume that  $X$  is countable. Due to Corollary 2.7, for all  $x \in X$ , we have

$$\lambda^{X \times \mathbb{N}}(\{\vartheta \in [0; 1]^{X \times \mathbb{N}} : x \text{ is } \vartheta\text{-stable}\}) = 1.$$

Using the countability of  $X$ , we obtain

$$\lambda^{X \times \mathbb{N}}(\{\vartheta \in [0; 1]^{X \times \mathbb{N}} : \forall x \in X (x \text{ is } \vartheta\text{-stable})\}) = 1.$$

Choose any particular  $\vartheta$  such that every  $x \in X$  is  $\vartheta$ -stable and let  $\mathcal{A}$  be any Moser–Tardos process with input  $\vartheta$ . Then  $\text{Stab}(\mathcal{A}) = X$ . Theorem 1.8 now follows from Proposition 2.3.

## 2.1 Moser–Tardos theory in the Borel setting

In this subsection we consider how Moser–Tardos theory can be adapted for the Borel setting.

Let  $X$  be a standard Borel space. Then  $[X]^{<\infty}$  is also equipped with a standard Borel structure. If, additionally,  $Y$  is a standard Borel space, then  $[X]_Y^{<\infty}$  is also a standard Borel space (indeed it can be identified with a Borel subset of  $[X \times Y]^{<\infty}$ ). An instance  $B$  over  $X$  is said to be *Borel* if it is Borel as a subset of  $[X]_{[0;1]}^{<\infty}$ . Note that in general the domain of a Borel instance is analytic.<sup>10</sup>

A map  $f: X' \rightarrow [0; 1]$  with  $X' \subseteq X$  is a *partial solution* for an instance  $B$  if for all  $S \in [X']^{<\infty}$ ,  $f|_S \notin B_S$ . A partial solution  $f$  is said to be *Borel* if it is a Borel function (in particular,  $\text{dom}(f)$  is Borel).

A Moser–Tardos process  $\mathcal{A} = (A_n)_{n=0}^\infty$  with Borel input  $\vartheta: X \rightarrow [0; 1]^\mathbb{N}$  is said to be *Borel* if for each  $n \in \mathbb{N}$ , the set  $A_n$  is Borel in  $[X]^{<\infty}$ . Note that if  $\mathcal{A}$  is a Borel Moser–Tardos process, then the associated maps  $t_n: X \rightarrow \mathbb{N}$  and  $f_n: X \rightarrow [0; 1]$  are also Borel. Now we have:

**Proposition 2.8** (Borel Moser–Tardos processes). *Let  $X$  be a standard Borel space and let  $B$  be a correct Borel instance over  $X$ . Let  $\vartheta: X \rightarrow [0; 1]^\mathbb{N}$  be Borel. Then there exists a Borel Moser–Tardos process  $\mathcal{A}$  with input  $\vartheta$ .*

We deduce Proposition 2.8 from the following lemma.

**Lemma 2.9** (Maximal disjoint subfamilies). *Let  $X$  be a standard Borel space and let  $A \subseteq [X]^{<\infty}$  be a Borel set such that for every  $x \in X$ , the set  $\{S \in A : x \in S\}$  is countable. Then there is a Borel maximal disjoint subset  $A_0 \subseteq A$ .*

*Proof.* This is a variant of the proof of [22, Lemma 7.3]. Define a Borel graph  $G$  on  $A$  by

$$S_1 G S_2 :\iff S_1 \neq S_2 \text{ and } S_1 \cap S_2 \neq \emptyset.$$

Note that, due to the lemma’s assumption, the graph  $G$  is locally countable. Let  $E$  denote the equivalence relation on  $X$  such that  $x E y$  if and only if there exists a sequence  $S_1, \dots$ ,

<sup>10</sup>In applications we usually have  $B_S \neq \emptyset \implies \lambda^S(B_S) > 0$ , which ensures the Borelness of  $\text{dom}(B)$  due to the “large section” uniformization theorem (see [20, Corollary 18.7]). However, the Borelness of  $\text{dom}(B)$  is not required for our arguments to work.

$S_k$ ,  $k \in \mathbb{N}$ , of elements of  $A$  such that  $x \in S_1$ ,  $y \in S_k$ , and for all  $1 \leq i < k$ ,  $S_i G S_{i+1}$ . The relation  $E$  is easily seen to be Borel, and, since  $G$  is locally countable, all the  $E$ -classes are countable. Therefore, by the Feldman–Moore theorem (see [38, Theorem 22.2]), there exists a countable sequence  $(\gamma_n)_{n=0}^\infty$  of Borel bijections  $\gamma_n: X \rightarrow X$  such that

$$xEy \iff \exists n \in \mathbb{N} (\gamma_n(x) = y).$$

Fix a Borel ordering  $<$  on  $X$  and let  $f: A \rightarrow [\mathbb{N}]^{<\infty}$  be a Borel function such that for all  $S \in A$ ,

$$S = \{\gamma_n(\min S) : n \in f(S)\}. \quad (2.4)$$

For instance,  $f(S)$  can be chosen to be the lexicographically least subset of  $\mathbb{N}$  with property (2.4). Note that any triple of the form  $(f(S), x, k)$ , where  $x \in S$ ,  $k \in f(S)$ , and  $\gamma_k(\min S) = x$ , uniquely determines  $S$ ; indeed, we have

$$S = \{\gamma_n(\gamma_k^{-1}(x)) : n \in f(S)\}.$$

Therefore, for each  $S \in A$ ,

$$|\{T \in G_S : f(T) = f(S)\}| \leq |S| |f(S)| - 1.$$

In particular, for each  $F \in [\mathbb{N}]^{<\infty}$ , the graph  $G^{(F)} := G|_{f^{-1}(F)}$  is locally finite. Hence, by Proposition 1.18, there is a Borel proper coloring  $c_F: f^{-1}(F) \rightarrow \mathbb{N}$  of  $G^{(F)}$ . Now let

$$c: A \rightarrow [\mathbb{N}]^{<\infty} \times \mathbb{N}: S \mapsto (f(S), c_{f(S)}(S)).$$

By construction,  $c$  is a Borel proper coloring of  $G$ . Thus,  $\chi_B(G) \leq \aleph_0$ . Given a countable Borel coloring of  $G$ , it is easy to construct a Borel maximal independent set in  $G$ , which is exactly what we want. ■

*Proof of Proposition 2.8.* Thanks to Lemma 2.9, it only remains to notice that for a Borel map  $f: X \rightarrow [0; 1]$ , the set

$$A' := \{S \in [X]^{<\infty} : f|_S \in B_S\}$$

is Borel. ■

We also have the following:

**Proposition 2.10.** *Let  $X$  be a standard Borel space and let  $B$  be a correct Borel instance over  $X$ . Let  $\mathcal{A}$  be a Borel Moser–Tardos process with Borel input  $\vartheta$ . Then the map*

$$f: \text{Stab}(\mathcal{A}) \rightarrow [0; 1]: x \mapsto \vartheta(x, \lim_{n \rightarrow \infty} t_n(x))$$

*is a Borel partial solution for  $B$ .* ■

### 3 Hereditarily finite sets

In this section we describe the construction of the “universal” combinatorial structure over a space  $X$ , whose points encode various combinatorial data that can be built from the elements of  $X$ .

For a set  $X$ , let  $F(X) := [X]^{<\infty} \cup \{\emptyset\}$  be the set of all finite subsets of  $X$ . The set  $\text{HF}(X)$  of all *hereditarily finite sets* over  $X$  is defined inductively as follows<sup>11</sup>:

- $\text{HF}^{(0)}(X) := X$ ;
- $\text{HF}^{(n+1)}(X) := \text{HF}^{(n)}(X) \cup F(\text{HF}^{(n)}(X))$  for all  $n \in \mathbb{N}$ ;
- $\text{HF}(X) := \bigcup_{n=0}^{\infty} \text{HF}^{(n)}(X)$  (note that this union is increasing).

In other words,  $\text{HF}(X)$  is the smallest superset of  $X$  that is closed under taking finite subsets. For  $h \in \text{HF}(X)$ , the *underlying set* of  $h$  (notation:  $\mathcal{U}(h)$ ) is the finite subset of  $X$  defined inductively by:

- for  $x \in X$ ,  $\mathcal{U}(x) := \{x\}$ ;
- for  $h \in \text{HF}^{(n+1)}(X) \setminus \text{HF}^{(n)}(X)$ ,  $\mathcal{U}(h) := \bigcup_{h' \in h} \mathcal{U}(h')$ .

Equivalently,  $\mathcal{U}(h)$  is the smallest subset of  $X$  such that  $h \in \text{HF}(\mathcal{U}(h))$ .

The *amplification* of  $X$  is the set

$$\text{HF}_0(X) := \{h \in \text{HF}(X) : \mathcal{U}(h) \neq \emptyset\}.$$

If  $X$  is a standard Borel space, then so are  $\text{HF}(X)$  and  $\text{HF}_0(X)$ . The space  $\text{HF}_0(X)$  fully encodes the “combinatorial structure” of  $X$ . For instance,  $\text{HF}_0(X)$  contains (as Borel subsets) the space  $X^{<\infty}$  of all nonempty finite sequences of elements of  $X$  and the space  $X \times \mathbb{N}$ , which can be viewed as the union of countably many disjoint copies of  $X$ . In fact,  $\text{HF}_0(X) \supseteq \text{HF}_0(X) \times \mathbb{N}$ , i.e.,  $\text{HF}_0(X)$  contains “countably many disjoint copies of itself.” If  $G$  is a Borel graph on  $X$ , then the edge set of  $G$ , i.e., the set

$$\mathcal{E}(G) := \{\{x, y\} : xGy\},$$

is also a Borel subset of  $\text{HF}_0(X)$ . So are other, more complicated, objects associated with  $G$ . For instance, the set of all cycles in  $G$ , i.e., the set of all finite subsets  $C \subseteq \mathcal{E}(G)$  whose elements form a cycle in  $G$ , is a Borel subset of  $\text{HF}_0(X)$ .

Note that if  $X' \subseteq \text{HF}_0(X)$  is Borel, then  $[X']^{<\infty}$  is a Borel subset of  $[\text{HF}_0(X)]^{<\infty}$  and  $[X']_{[0;1]}^{<\infty}$  is a Borel subset of  $[\text{HF}_0(X)]_{[0;1]}^{<\infty}$ . Therefore, a Borel instance  $B$  over  $X'$  is also a Borel instance over  $\text{HF}_0(X)$ . Because of that, we will be only considering instances over  $\text{HF}_0(X)$ , and this will include various combinatorial applications such as vertex coloring or edge coloring.

Functions between sets can be naturally extended to functions between their amplifications. Namely, for a map  $\varphi: X \rightarrow Y$ , define the map  $\text{HF}(\varphi): \text{HF}(X) \rightarrow \text{HF}(Y)$  inductively by:

---

<sup>11</sup>Here we view the points of  $X$  themselves as *urelements*, i.e., not sets. Formally, we can replace  $X$  with, say, the diagonal  $\Delta_X^{\mathbb{N}} := \{(x, x, x, \dots) : x \in X\}$  in  $X^{\mathbb{N}}$ , ensuring that no point in  $X$  is a finite set.

- for  $x \in X$ ,  $\text{HF}(\varphi)(x) := \varphi(x)$ ;
- for  $h \in \text{HF}^{(n+1)}(X) \setminus \text{HF}^{(n)}(X)$ ,  $\text{HF}(\varphi)(h) := \{\text{HF}(\varphi)(h') : h' \in h\}$ .

The *amplification* of  $\varphi$  is the map  $\tilde{\varphi} := \text{HF}(\varphi)|_{\text{HF}_0(X)}$ . Note that the range of  $\tilde{\varphi}$  is contained in  $\text{HF}_0(Y)$ . For  $S \in [X]^{<\infty}$ , we have  $\tilde{\varphi}(S) = \varphi(S)$  (where  $\varphi(S)$  denotes, as usual, the image of  $S$  under  $\varphi$ ). If  $\varphi$  is injective (or surjective), then so is  $\tilde{\varphi}$ . If  $X$  and  $Y$  are standard Borel spaces and  $\varphi$  is Borel, then so is  $\tilde{\varphi}$ .

## 4 The approximate Borel LLL

In this section we state and prove our first main result: an approximate measurable version of the LLL for Borel instances.

Let  $X$  be a standard Borel space and let  $\mu \in \mathcal{P}(X)$ . Suppose that  $B$  is a Borel instance over  $\text{HF}_0(X)$  and let  $f: \text{HF}_0(X) \rightarrow [0; 1]$  be a Borel map. We want to use  $\mu$  to measure how well  $f$  solves  $B$ . Since  $\mu$  is concentrated on  $X$ , it is reasonable to consider, for each  $x \in X$ , the following set:

$$\partial_x(B) := \{S \in \text{dom}(B) : \exists h \in S(x \in \mathcal{U}(h))\}.$$

We call  $\partial_x(B)$  the *shadow* of  $B$  at  $x$ . An instance  $B$  over  $\text{HF}_0(X)$  is said to be *hereditarily locally finite* if  $\partial_x(B)$  is finite for all  $x \in X$ . Now, for a Borel map  $f: \text{HF}_0(X) \rightarrow [0; 1]$ , let its *B-defect* be the set

$$D(f, B) := \{x \in X : \exists S \in \partial_x(B)(f|_S \in B_S)\}.$$

Note that if  $B$  is hereditarily locally finite, then the set  $D(f, B) \subseteq X$  is Borel.

Now we are ready to state our first main result.

**Theorem 4.1** (The approximate Borel LLL). *Let  $X$  be a standard Borel space and let  $\mu \in \mathcal{P}(X)$ . Let  $B$  be a hereditarily locally finite correct Borel instance over  $\text{HF}_0(X)$ . Then for any  $\varepsilon > 0$ , there is a Borel map  $f: \text{HF}_0(X) \rightarrow [0; 1]$  such that  $\mu(D(f, B)) < \varepsilon$ .*

### 4.1 Proof of Theorem 4.1

Let  $X$  be a standard Borel space, let  $\mu \in \mathcal{P}(X)$ , and let  $B$  be a hereditarily locally finite correct Borel instance over  $\text{HF}_0(X)$ . Fix some  $\varepsilon > 0$ . For  $S \in \text{dom}(B)$  and  $n \in \mathbb{N}$ , let  $\mathfrak{P}_n(S) \subseteq \mathfrak{P}(S)$  denote the set of all  $\mathcal{P} \in \mathfrak{P}(S)$  with  $h(\mathcal{P}) = n + 1$ . In particular,

$$\mathfrak{P}(S) = \bigcup_{n=0}^{\infty} \mathfrak{P}_n(S), \tag{4.1}$$

and the union on the right-hand side of (4.1) is disjoint. For  $n \in \mathbb{N}$ , let

$$D_n := \left\{ x \in X : \sum_{S \in \partial_x(B)} \sum_{\mathcal{P} \in \mathfrak{P}_n(S)} \lambda^{\text{supp}(\mathcal{P}) \times \mathbb{N}}(W(\mathcal{P})) > \frac{\varepsilon}{2} \right\}.$$



It is clear from the above definition that for all  $n \in \mathbb{N}$ , the set  $D_n$  is analytic (recall that the set  $\text{dom}(B)$  is analytic). This in particular implies that  $D_n$  is  $\mu$ -measurable, which will be sufficient for our purposes. However, one can show that  $D_n$  is, in fact, Borel. Indeed, notice that if there is  $\tau \in \mathcal{P}$  with  $\lambda^{\text{dom}(\tau)}(B_{\text{dom}(\tau)}) = 0$ , then  $\lambda^{\text{supp}(\mathcal{P}) \times \mathbb{N}}(W(\mathcal{P})) = 0$ , and the set

$$\{S \in [X]^{<\infty} : \lambda^S(B_S) > 0\} \subseteq \text{dom}(B)$$

is Borel due to the “large section” uniformization theorem (see [20, Corollary 18.7]).

Due to Corollary 2.6 and the fact that  $B$  is hereditarily locally finite, for every  $x \in X$ , we have

$$\sum_{S \in \partial_x(B)} \sum_{\mathcal{P} \in \mathfrak{P}(S)} \lambda^{\text{supp}(\mathcal{P}) \times \mathbb{N}}(W(\mathcal{P})) < \infty.$$

Hence we can choose  $K \in \mathbb{N}$  large enough so that  $\mu(D_K) \leq \varepsilon/2$ .

Let  $G$  be the graph on  $\text{HF}_0(X)$  given by:

$$h_1 G h_2 : \iff h_1 \neq h_2 \text{ and } \exists S \in \text{dom}(B)(h_1 \in S \text{ and } h_2 \in S).$$

In other words,  $h_1 G h_2$  if and only if  $h_1 \neq h_2$  and  $\text{dom}_{h_1}(B) \cap \text{dom}_{h_2}(B) \neq \emptyset$ . Clearly,  $G$  is analytic, and the hereditary local finiteness of  $B$  implies that  $G$  is locally finite. For  $n \in \mathbb{N}$ , let  $G^n$  be the analytic graph on  $\text{HF}_0(X)$  such that  $h_1 G^n h_2$  if and only if  $h_1 \neq h_2$  and the distance between  $h_1$  and  $h_2$  in  $G$  is at most  $n$  (in particular,  $G^1 = G$ ). Since  $G$  is locally finite, so is  $G^n$  for each  $n \in \mathbb{N}$ . By Proposition 1.18,  $\chi_B(G^n) \leq \aleph_0$  for all  $n \in \mathbb{N}$ , so let  $c: \text{HF}_0(X) \rightarrow \mathbb{N}$  be a Borel proper coloring of  $G^{2(K+1)}$ .

For a function  $\vartheta: \mathbb{N} \rightarrow [0; 1]^{\mathbb{N}}$ , we can consider the composition  $\vartheta \circ c: \text{HF}_0(X) \rightarrow [0; 1]^{\mathbb{N}}$ , which is a Borel table (in the sense of the Moser–Tardos algorithm on  $\text{HF}_0(X)$ ). Let  $Q \subseteq X \times [0; 1]^{\mathbb{N} \times \mathbb{N}}$  be the set defined as follows:

$$(x, \vartheta) \in Q : \iff \exists S \in \partial_x(B) \exists \mathcal{P} \in \mathfrak{P}_K(S) (\mathcal{P} \text{ appears in } \vartheta \circ c).$$

By definition,  $Q$  is analytic; in particular, it is  $(\mu \times \lambda^{\mathbb{N} \times \mathbb{N}})$ -measurable. Again, one can show that  $Q$  is, in fact, Borel, since the set

$$\{\tau: S \rightarrow \mathbb{N} : S \in [X]^{<\infty} \text{ and } (\vartheta \circ c)(\tau) \in B_S\}$$

is Borel.

**Lemma 4.2.** *For every  $x \in X \setminus D_K$ , we have  $\lambda^{\mathbb{N} \times \mathbb{N}}(Q_x) \leq \varepsilon/2$ .*

*Proof.* It is easy to see that for any neat pile  $\mathcal{P}$  with a unique top element  $\tau$  and for any  $\tau' \in \mathcal{P}$ , there is a sequence  $\tau_0, \dots, \tau_k \in \mathcal{P}$  with  $k \in \mathbb{N}$  such that  $\tau_0 = \tau'$ ,  $\tau_k = \tau$ , and  $\tau_0 \prec \tau_1 \prec \dots \prec \tau_k$ . In particular, we have  $\text{dom}(\tau_i) \cap \text{dom}(\tau_{i+1}) \neq \emptyset$  for all  $0 \leq i < k$ , so the distance in  $G$  between any element of  $\text{dom}(\tau')$  and any element of  $\text{dom}(\tau)$  is at most  $k + 1 \leq h(\mathcal{P})$ . Therefore, the distance in  $G$  between any two elements of  $\text{supp}(\mathcal{P})$  is at most  $2h(\mathcal{P})$ .

Now let  $x \in X \setminus D_K$ , let  $S \in \partial_x(B)$ , and let  $\mathcal{P} \in \mathfrak{P}_K(S)$ . Since  $h(\mathcal{P}) = K + 1$ , the distance in  $G$  between any two elements of  $\text{supp}(\mathcal{P})$  is at most  $2(K + 1)$ ; in other words, any

two distinct elements of  $\text{supp}(\mathcal{P})$  are adjacent in  $G^{2(K+1)}$ . Therefore,  $c|_{\text{supp}(\mathcal{P})}$  is injective. Hence  $c|_{\text{supp}(\mathcal{P})}$  induces a measure-preserving bijection

$$\varphi: [0; 1]^{c(\text{supp}(\mathcal{P})) \times \mathbb{N}} \rightarrow [0; 1]^{\text{supp}(\mathcal{P}) \times \mathbb{N}}: \eta \mapsto \eta \circ (c|_{\text{supp}(\mathcal{P})}).$$

Thus, if we let

$$\pi: [0; 1]^{\mathbb{N} \times \mathbb{N}} \rightarrow [0; 1]^{c(\text{supp}(\mathcal{P})) \times \mathbb{N}}: \vartheta \mapsto \vartheta|_{c(\text{supp}(\mathcal{P}))},$$

then the map

$$\varphi \circ \pi: [0; 1]^{\mathbb{N} \times \mathbb{N}} \rightarrow [0; 1]^{\text{supp}(\mathcal{P}) \times \mathbb{N}}: \vartheta \mapsto (\vartheta \circ c)|_{\text{supp}(\mathcal{P})}$$

is measure-preserving. Since

$$\mathcal{P} \text{ appears in } \vartheta \circ c \iff (\vartheta \circ c)|_{\text{supp}(\mathcal{P})} \in W(\mathcal{P}),$$

we now have

$$\lambda^{\mathbb{N} \times \mathbb{N}}(\{\vartheta \in [0; 1]^{\mathbb{N} \times \mathbb{N}} : \mathcal{P} \text{ appears in } \vartheta \circ c\}) = \lambda^{\text{supp}(\mathcal{P}) \times \mathbb{N}}(W(\mathcal{P})).$$

Therefore,

$$\begin{aligned} \lambda^{\mathbb{N} \times \mathbb{N}}(Q_x) &\leq \sum_{S \in \partial_x(B)} \sum_{\mathcal{P} \in \mathfrak{P}_K(S)} \lambda^{\mathbb{N} \times \mathbb{N}}(\{\vartheta \in [0; 1]^{\mathbb{N} \times \mathbb{N}} : \mathcal{P} \text{ appears in } \vartheta \circ c\}) \\ &= \sum_{S \in \partial_x(B)} \sum_{\mathcal{P} \in \mathfrak{P}_K(S)} \lambda^{\text{supp}(\mathcal{P}) \times \mathbb{N}}(W(\mathcal{P})) \leq \frac{\varepsilon}{2}, \end{aligned}$$

by the definition of  $D_K$ . ■

Using Lemma 4.2, we get

$$(\mu \times \lambda^{\mathbb{N} \times \mathbb{N}})(Q) = \int_X \lambda^{\mathbb{N} \times \mathbb{N}}(Q_x) d\mu(x) \leq \mu(D_K) + (1 - \mu(D_K)) \cdot \frac{\varepsilon}{2} < \varepsilon.$$

This in particular means that for some  $\vartheta \in [0; 1]^{\mathbb{N} \times \mathbb{N}}$ , we have  $\mu(Q^\vartheta) < \varepsilon$ . Fix any such  $\vartheta$  and let  $\mathcal{A} = (A_n)_{n=0}^\infty$  be a Borel Moser–Tardos process with input  $\vartheta \circ c$ . Let  $t_n: X \rightarrow \mathbb{N}$  and  $f_n: X \rightarrow [0; 1]$ , for  $n \in \mathbb{N}$ , denote the associated maps.

**Lemma 4.3.**  $D(f_K, B) \subseteq Q^\vartheta$ .

*Proof.* Indeed, by Proposition 2.4, if  $S \in A'_K$ , then there is a neat pile  $\mathcal{P} \in \mathfrak{P}_K(S)$  such that  $\mathcal{P}$  appears in  $\vartheta \circ c$ . But if  $x \in D(f_K, B)$ , then there is  $S \in \partial_x(B)$  such that  $f_K|_S \in B_S$ , i.e.,  $S \in A'_K$ . Therefore,  $(x, \vartheta) \in Q$ , as desired. ■

Finally, if we take  $f := f_K$ , then

$$\mu(D(f, B)) = \mu(D(f_K, B)) \leq \mu(Q^\vartheta) < \varepsilon,$$

and the proof of Theorem 4.1 is complete.

## 5 The LLL for probability measure-preserving group actions

In this section we introduce the notion of an isomorphism structure on an equivalence relation. As we discussed in the introduction, this notion is designed to capture the fact that the instances of the LLL that appear in actual combinatorial applications are invariant under the automorphisms of the underlying combinatorial object (for example, a graph). We then define the LLL game, which models the process of proving a combinatorial result by applying the LLL iteratively. These definitions will allow us to state the measurable version of the LLL for probability measure-preserving actions of countable groups that admit factor maps to  $[0; 1]$ -shift actions.

### 5.1 Definitions and the statement of the theorem

#### 5.1.1 Equivalence relations

Let  $E$  be an equivalence relation on a set  $X$ . A subset  $X' \subseteq X$  is *E-invariant* if for all  $x \in X'$ ,  $y \in X$ ,  $xEy \implies y \in X'$ . For  $\emptyset \neq S \subseteq X$ , let  $[S]_E$  denote the *E-saturation* of  $S$ , i.e., the smallest invariant subset of  $X$  that contains  $S$ . For  $x \in X$ , let  $[x]_E := [\{x\}]_E$ . The *restriction* of  $E$  to a subset  $X' \subseteq X$  is the equivalence relation  $E|_{X'} := E \cap (X')^2$  on  $X'$ .

An equivalence relation  $E$  is said to be *countable* if every  $E$ -class is countable. Note that if  $X$  is a standard Borel space and  $E$  is a countable Borel (as a subset of  $X^2$ ) equivalence relation on  $X$ , then for every Borel  $S \subseteq X$ ,  $[S]_E$  is also Borel.

For a graph  $G$  with vertex set  $X$ ,  $E_G$  denotes the equivalence relation on  $X$  whose classes are the connected components of  $G$ . Note that if  $X$  is a standard Borel space and  $G$  is a locally countable Borel graph on  $X$ , then  $E_G$  is a countable Borel equivalence relation.

For a set  $X$  and an equivalence relation  $E$  on  $X$ , let  $[E]^{<\infty}$  be the subset of  $[X]^{<\infty}$  defined by

$$S \in [E]^{<\infty} :\iff \forall x, y \in S (xEy).$$

For a set  $Y$ , let  $[E]_Y^{<\infty}$  denote the subset of  $[X]_Y^{<\infty}$  consisting of the functions  $f: S \rightarrow Y$  with  $S \in [E]^{<\infty}$ . Let

$$\text{HF}_0(E) := \{h \in \text{HF}_0(X) : \mathcal{U}(h) \in [E]^{<\infty}\}.$$

The *amplification* of  $E$  is the equivalence relation  $\tilde{E}$  on  $\text{HF}_0(E)$  given by

$$h_1 \tilde{E} h_2 :\iff [\mathcal{U}(h_1)]_E = [\mathcal{U}(h_2)]_E.$$

Note that  $X$  is a complete section for  $\tilde{E}$  (i.e.,  $X$  intersects every  $\tilde{E}$ -class) and  $\tilde{E}|_X = E$ .

If  $E$  is an equivalence relation on  $X$ ,  $F$  is an equivalence relation on  $Y$ , and  $\varphi: X \rightarrow Y$  is a map such that  $xEy \implies \varphi(x)E\varphi(y)$ , then for the map  $\tilde{\varphi}: \text{HF}_0(X) \rightarrow \text{HF}_0(Y)$ , we have  $\tilde{\varphi}(\text{HF}_0(E)) \subseteq \text{HF}_0(F)$ , i.e., the range of the restriction  $\tilde{\varphi}|_{\text{HF}_0(E)}$  is contained in  $\text{HF}_0(F)$ .

If  $X$  and  $Y$  are standard Borel spaces and  $E$  is a countable Borel equivalence relation on  $X$ , then  $[E]^{<\infty}$  is a Borel subset of  $[X]^{<\infty}$ ,  $[E]_Y^{<\infty}$  is a Borel subset of  $[X]_Y^{<\infty}$ ,  $\text{HF}_0(E)$  is a Borel subset of  $\text{HF}_0(X)$ , and  $\tilde{E}$  is a countable Borel equivalence relation on  $\text{HF}_0(E)$ .

### 5.1.2 Isomorphism structures

If  $E$  is an equivalence relation on a set  $X$ , then a collection  $\mathcal{I}$  of bijections of the form  $\varphi: X_1 \rightarrow X_2$ , where  $X_1, X_2 \subseteq X$ , is an *isomorphism structure* on  $E$  if the following conditions are satisfied:

1.  $\mathcal{I}$  is a groupoid<sup>12</sup> with the set of objects  $\text{dom}(\mathcal{I}) := \{\text{dom}(\varphi) : \varphi \in \mathcal{I}\}$ ; more precisely:
  - (a) for all  $S \in \text{dom}(\mathcal{I})$ ,  $\text{id}_S \in \mathcal{I}$ ;
  - (b) for all  $\varphi \in \mathcal{I}$ ,  $\varphi^{-1} \in \mathcal{I}$ ;
  - (c) for all  $\varphi_1, \varphi_2 \in \mathcal{I}$ , if  $\text{im}(\varphi_2) = \text{dom}(\varphi_1)$ , then  $\varphi_1 \circ \varphi_2 \in \mathcal{I}$ ;
2. each  $S \in \text{dom}(\mathcal{I})$  is contained in a single  $E$ -class;
3. for every  $S \in [E]^{<\infty}$ , there is  $S' \in \text{dom}(\mathcal{I})$  such that  $S \subseteq S'$ .

For an isomorphism structure  $\mathcal{I}$  on  $E$ , let

$$\mathcal{I}^{<\infty} := \{\varphi|S : \varphi \in \mathcal{I} \text{ and } S \in [\text{dom}(\varphi)]^{<\infty}\}.$$

The last condition in the definition of an isomorphism structure is needed to ensure that for every  $S \in [E]^{<\infty}$ ,  $\text{id}_S \in \mathcal{I}^{<\infty}$ .

We will consider the following main examples of isomorphism structures.

**Example 5.1** (Isomorphism structures induced by graphs). Let  $G$  be a graph with vertex set  $X$ . We define the isomorphism structure  $\mathcal{I}_G$  on  $E_G$  as follows: A bijection  $\varphi: C_1 \rightarrow C_2$  belongs to  $\mathcal{I}_G$  if and only if  $C_1$  and  $C_2$  are connected components of  $G$  and  $\varphi$  is an isomorphism between  $G|C_1$  and  $G|C_2$ . We can also define the isomorphism structure  $\mathcal{I}_G^*$  such that a bijection  $\varphi: S_1 \rightarrow S_2$  belongs to  $\mathcal{I}_G^*$  if and only if  $S_1, S_2 \in [E_G]^{<\infty}$  and  $\varphi$  is an isomorphism between  $G|S_1$  and  $G|S_2$ . Note that  $\mathcal{I}_G^{<\infty} \subseteq \mathcal{I}_G^*$ . The structure  $\mathcal{I}_G^*$  will play a prominent role in Section 6.

**Example 5.2** (Isomorphism structures induced by group actions). Let  $a: \Gamma \curvearrowright X$  be an action of a group  $\Gamma$  on a set  $X$ . Let  $E_a$  denote the corresponding orbit equivalence relation. The isomorphism structure  $\mathcal{I}_a$  on  $E_a$  is defined as follows: A bijection  $\varphi: C_1 \rightarrow C_2$  belongs to  $\mathcal{I}_a$  if and only if  $C_1$  and  $C_2$  are  $E_a$ -classes and  $\varphi$  is  $\Gamma$ -equivariant, i.e., for all  $x \in C_1$  and  $\gamma \in \Gamma$ ,  $\varphi(\gamma \cdot x) = \gamma \cdot \varphi(x)$ . Note that if  $a: \Gamma \curvearrowright X$  is an action of a group  $\Gamma$  on a set  $X$  and  $S \subseteq \Gamma$  is a generating set, then  $\mathcal{I}_a \subseteq \mathcal{I}_{G(a,S)}$ .

If  $\mathcal{I}$  is an isomorphism structure on  $E$ , where  $E$  is an equivalence relation on  $X$ , then the *amplification* of  $\mathcal{I}$  is the set  $\tilde{\mathcal{I}} := \{\tilde{\varphi} : \varphi \in \mathcal{I}\}$ . It is easy to verify that  $\tilde{\mathcal{I}}$  is an isomorphism structure on  $\tilde{E}$ . For a map  $f: X \rightarrow Y$ , where  $Y$  is a set, the *expansion* of  $\mathcal{I}$  by  $f$  is the subset  $\mathcal{I}[f] \subseteq \mathcal{I}$  defined as follows:

$$\mathcal{I}[f] := \{\varphi \in \mathcal{I} : f(x) = f(\varphi(x)) \text{ for all } x \in \text{dom}(\varphi)\}.$$

---

<sup>12</sup>A *groupoid* is a category in which every morphism has an inverse.

Note that  $\mathcal{I}[f]$  is also an isomorphism structure on  $E$ . For a map  $f: X' \rightarrow Y$ , defined on a subset  $X' \subseteq X$ , let  $\mathcal{I}[f] := \mathcal{I}[g]$ , where  $g: X \rightarrow Y \cup \{z\}$  with  $z \notin Y$  is given by

$$g(x) := \begin{cases} f(x) & \text{if } x \in X'; \\ z & \text{otherwise.} \end{cases}$$

The name “expansion” conveys the following intuition: If  $\mathcal{I}$  is a family of isomorphisms between certain substructures of  $X$  (for example, as in the case of  $\mathcal{I}_G$ , between connected components of a graph  $G$  on  $X$ ), then expanding  $\mathcal{I}$  by  $f$  corresponds to adding  $f$  to  $X$  as a new “predicate,” whose values have to be preserved by the isomorphisms between substructures of the expanded structure.

### 5.1.3 Invariant instances

An *instance (of the LLL)* over a countable equivalence relation  $E$  on a set  $X$  is a subset  $B \subseteq [E]_{[0;1]}^{<\infty}$  such that for each  $S \in [E]^{<\infty}$ , the set  $B_S$  is Borel. Since  $[E]_{[0;1]}^{<\infty} \subseteq [X]_{[0;1]}^{<\infty}$ , any instance over  $E$  is, in particular, an instance over  $X$ , so we can apply to it all the notions defined previously for instances over  $X$ , such as Borelness or correctness. Note that every instance over  $E$  is guaranteed to be locally countable, since every  $E$ -class is countable.

If  $X$  is a standard Borel space,  $E$  is a countable Borel equivalence relation on  $X$ ,  $B$  is a Borel instance over  $E$ , and  $\mu \in \mathbf{P}(X)$ , then a  $\mu$ -*solution* for  $B$  is a Borel partial solution  $f$  for  $B$  such that  $\text{dom}(f)$  is an  $E$ -invariant  $\mu$ -conull Borel subset of  $X$ .

An instance  $B$  over  $X$  is said to be  $\varphi$ -*invariant*, where  $\varphi$  is a map with  $\text{dom}(\varphi), \text{im}(\varphi) \in [X]^{<\infty}$ , if for all  $f: \text{im}(\varphi) \rightarrow [0;1]$ ,

$$f \in B_{\text{im}(\varphi)} \iff f \circ \varphi \in B_{\text{dom}(\varphi)}.$$

Let  $\mathcal{I}$  be an isomorphism structure on  $E$ . An instance  $B$  over  $E$  is said to be  $\mathcal{I}$ -*invariant on  $X'$* , where  $X' \subseteq X$ , if it is  $\varphi$ -invariant for all  $\varphi \in \mathcal{I}^{<\infty}$  with  $\text{dom}(\varphi), \text{im}(\varphi) \in [X']^{<\infty}$ . If  $\mu \in \mathbf{P}(X)$ , then  $B$  is said to be  $\mathcal{I}$ -*invariant  $\mu$ -a.e.* if it is  $\mathcal{I}$ -invariant on an  $E$ -invariant  $\mu$ -conull Borel subset of  $X$ .

### 5.1.4 The LLL game

An *L-system*<sup>13</sup> is a tuple  $\mathbf{L} = (X, E, \mathcal{I}, \mu)$ , where  $X$  is a standard Borel space,  $E$  is a countable Borel equivalence relation on  $X$ ,  $\mathcal{I}$  is an isomorphism structure on  $E$ , and  $\mu \in \mathbf{P}(X)$ . An *instance (of the LLL)* over  $\mathbf{L}$  is a  $\mathcal{I}$ -invariant  $\mu$ -a.e. instance over  $E$ . The *amplification* of  $\mathbf{L}$  is the L-system  $\text{HF}_0(\mathbf{L}) := (\text{HF}_0(E), \tilde{E}, \tilde{\mathcal{I}}, \mu)$  (so the measure in  $\text{HF}_0(\mathbf{L})$  is concentrated on  $X$ ). For a Borel map  $f: X' \rightarrow Y$ , where  $X' \subseteq X$  is Borel and  $Y$  is a standard Borel space, the *expansion* of  $\mathbf{L}$  by  $f$  is the L-system  $\mathbf{L}[f] := (X, E, \mathcal{I}[f], \mu)$ . For a given L-system  $\mathbf{L}$ , we refer to its components as  $X_{\mathbf{L}}$ ,  $E_{\mathbf{L}}$ ,  $\mathcal{I}_{\mathbf{L}}$ , and  $\mu_{\mathbf{L}}$ .

As we mentioned in the introduction, many combinatorial arguments contain iterated applications of the LLL, where the output of a previous iteration can be used to create an instance for the next one. To accommodate such arguments, we introduce the following definition.

---

<sup>13</sup>“L” stands for “Lovász.”

**Definition 5.3** (The LLL game). The *LLL game* on an L-system  $\mathbf{L}$  is played as follows. Let  $\mathbf{L}_0 := \mathbf{L}$ . On Step  $k$ , where  $k \in \mathbb{N}$ , Player I chooses a correct Borel instance  $B_k$  over  $\mathbf{L}_k$ . If  $B_k$  has no  $\mu_{\mathbf{L}}$ -solution, then Player II loses; otherwise, he chooses a  $\mu_{\mathbf{L}}$ -solution  $f_k$  for  $B_k$  and sets  $\mathbf{L}_{k+1} := \mathbf{L}_k[f_k]$ . Player II wins if he does not lose on any finite step of the game.

We say that an L-system  $\mathbf{L}$  *satisfies the LLL* if Player II has a winning strategy in the LLL game on  $\text{HF}_0(\mathbf{L})$ .

For a Borel action  $a: \Gamma \curvearrowright X$  of a countable group  $\Gamma$  on a standard Borel space  $X$  and  $\mu \in \mathcal{P}(X)$ , let  $\mathbf{L}(a, \mu)$  denote the L-system  $(X, E_a, \mathcal{I}_a, \mu)$ . Now we are ready to state our second main result.

**Theorem 5.4** (The measurable LLL for group actions). *Suppose that  $a: \Gamma \curvearrowright X$  is a Borel action of a countable group  $\Gamma$  on a standard Borel space  $X$  and let  $\mu \in \mathcal{P}(X)$  be an invariant probability measure. If there exists a factor map  $\pi: (X, \mu) \rightarrow ([0; 1]^\Gamma, \lambda^\Gamma)$  to the shift action of  $\Gamma$  on  $[0; 1]^\Gamma$ , then  $\mathbf{L}(a, \mu)$  satisfies the LLL.*

## 5.2 Proof of Theorem 5.4

### 5.2.1 Outline of the proof

Let  $\mathcal{G}$  denote the class of all L-systems of the form  $\mathbf{L}(a, \mu)$ , where  $a: \Gamma \curvearrowright X$  is a Borel action of a countable group  $\Gamma$  on a standard Borel space  $X$ ,  $\mu \in \mathcal{P}(X)$ , and there is a factor map  $\pi: (X, \mu) \rightarrow ([0; 1]^\Gamma, \lambda^\Gamma)$ . Let  $\mathcal{L}$  be the class of all L-systems that satisfy the LLL. We want to show that  $\mathcal{G} \subseteq \mathcal{L}$ . To do this, we will introduce an intermediate class  $\mathcal{C}$  such that  $\mathcal{G} \subseteq \mathcal{C} \subseteq \mathcal{L}$ .

Our strategy for showing that  $\mathcal{C} \subseteq \mathcal{L}$  will be to ensure that  $\mathcal{C}$  has the following two properties:

- (A1) if  $\mathbf{L} \in \mathcal{C}$ , then  $\text{HF}_0(\mathbf{L}) \in \mathcal{C}$ ;
- (A2) if  $\mathbf{L} \in \mathcal{C}$  and  $B$  is a correct Borel instance over  $\mathbf{L}$ , then there exists a  $\mu_{\mathbf{L}}$ -solution  $f$  for  $B$  such that  $\mathbf{L}[f] \in \mathcal{C}$ .

The above conditions imply that every L-system in  $\mathcal{C}$  satisfies the LLL. Indeed, due to Property (A1), it is enough to show that for every  $\mathbf{L} \in \mathcal{C}$ , Player II has a winning strategy in the LLL game on  $\mathbf{L}$ . The existence of such a strategy is guaranteed by Property (A2), since, provided that  $\mathbf{L}_k \in \mathcal{C}$ , Player II can always find a  $\mu_{\mathbf{L}}$ -solution  $f_k$  for  $B_k$  such that  $\mathbf{L}_{k+1} = \mathbf{L}_k[f_k] \in \mathcal{C}$ .

It is easy to see that Property (A1) fails for  $\mathcal{G}$ ; in fact, if  $\mathbf{L} \in \mathcal{G}$ , then  $\mu_{\mathbf{L}}$  is  $E_{\mathbf{L}}$ -invariant, while it is not even  $\tilde{E}_{\mathbf{L}}$ -quasi-invariant. To overcome this difficulty, we will introduce *countable Borel groupoids*—algebraic structures more general than countable groups—and their actions on Borel spaces. As it will turn out, every Borel action of a countable Borel groupoid on a standard probability space induces an L-system, and its definition extends the one for actions of countable groups. Also, one can define shift actions of countable Borel groupoids, generalizing shift actions of countable groups. Our choice for  $\mathcal{C}$  will be the class of all L-systems that admit factor maps (we will define what a factor map between two general L-systems is in §5.2.2) to L-systems induced by shift actions of countable Borel groupoids.

### 5.2.2 Factors of L-systems

In this section we introduce the notion of a factor map between two L-systems. It will allow us to transfer instances of the LLL from a given L-system to a simpler or better-behaved one.

**Definition 5.5** (Factors). Let  $\mathbf{L}_1 = (X_1, E_1, \mathcal{I}_1, \mu_1)$  and  $\mathbf{L}_2 = (X_2, E_2, \mathcal{I}_2, \mu_2)$  be L-systems. A Borel map  $\pi: X'_1 \rightarrow X_2$ , defined on an  $E_1$ -invariant  $\mu_1$ -conull Borel subset  $X'_1 \subseteq X_1$ , is a *factor map* (notation:  $\pi: \mathbf{L}_1 \rightarrow \mathbf{L}_2$ ) if the following conditions hold:

1.  $\pi_*(\mu_1) = \mu_2$ ;
2. for each  $E_1$ -class  $C \subseteq X'_1$ ,  $\pi(C)$  is an  $E_2$ -class and  $\pi|_C$  is injective;
3. for all  $S_1, S_2 \in [E_1|X'_1]^{<\infty}$ , if  $\varphi_2 \in \mathcal{I}_2^{<\infty}$  is a bijection between  $\pi(S_1)$  and  $\pi(S_2)$ , then there is a bijection  $\varphi_1 \in \mathcal{I}_1^{<\infty}$  between  $S_1$  and  $S_2$  that makes the following diagram commute:

$$\begin{array}{ccc} S_1 & \overset{\varphi_1}{\dashrightarrow} & S_2 \\ \downarrow \pi & & \downarrow \pi \\ \pi(S_1) & \xrightarrow{\varphi_2} & \pi(S_2). \end{array}$$

**Proposition 5.6.** *Let  $\mathbf{L}_1$  and  $\mathbf{L}_2$  be L-systems. Suppose that there exists a factor map  $\pi: \mathbf{L}_1 \rightarrow \mathbf{L}_2$ . Then there exists a factor map from  $\text{HF}_0(\mathbf{L}_1)$  to  $\text{HF}_0(\mathbf{L}_2)$ .*

*Proof.* For  $k \in \{1, 2\}$ , let

$$X_k := X_{\mathbf{L}_k} \quad \text{and} \quad E_k := E_{\mathbf{L}_k}.$$

Let  $X'_1 := \text{dom}(\pi)$  and let  $\tilde{\pi}: \text{HF}_0(X'_1) \rightarrow \text{HF}_0(X_2)$  be the amplification of  $\pi$ . Then  $\tilde{\pi}|_{\text{HF}_0(E_1|X'_1)}$  is easily seen to be a factor map between  $\text{HF}_0(\mathbf{L}_1)$  and  $\text{HF}_0(\mathbf{L}_2)$ .  $\blacksquare$

**Lemma 5.7.** *Let  $\mathbf{L}_1$  and  $\mathbf{L}_2$  be L-systems. Suppose that there exists a factor map  $\pi: \mathbf{L}_1 \rightarrow \mathbf{L}_2$ . Then for any correct Borel instance  $B_1$  over  $\mathbf{L}_1$ , there exists a correct Borel instance  $B_2$  over  $\mathbf{L}_2$  such that if  $f$  is a  $\mu_{\mathbf{L}_2}$ -solution for  $B_2$ , then  $f \circ \pi$ , possibly restricted to a smaller invariant conull Borel subset of  $X_{\mathbf{L}_1}$ , is a  $\mu_{\mathbf{L}_1}$ -solution for  $B_1$ .*

*Proof.* For  $k \in \{1, 2\}$ , let

$$\mathbf{L}_k =: (X_k, E_k, \mathcal{I}_k, \mu_k).$$

Let  $B_1$  be a correct Borel instance over  $\mathbf{L}_1$ . Restricting the domain of  $\pi$  to a smaller  $E_1$ -invariant  $\mu_1$ -conull Borel subset of  $X_1$ , we can arrange that  $B_1$  is  $\mathcal{I}_1$ -invariant on  $\text{dom}(\pi)$ ,  $\text{dom}(B_1) \subseteq [E_1|\text{dom}(\pi)]^{<\infty}$ , and  $\text{im}(\pi)$  is Borel in  $X_2$ . Let  $X'_1 := \text{dom}(\pi)$ ,  $X'_2 := \text{im}(\pi)$ . Since  $\pi$  is a factor map,  $X'_1$  and  $X'_2$  are invariant conull subsets of  $X_1$  and  $X_2$ , respectively.

For  $T \in [E_2|X'_2]^{<\infty}$ , choose any  $S \in [E_1|X'_1]^{<\infty}$  such that  $\pi(S) = T$  and define

$$(B_2)_T := \{f \in [0; 1]^T : f \circ (\pi|_S) \in (B_1)_S\}.$$

We claim that this definition does not depend on the choice of  $S$ . Indeed, suppose that  $S_1, S_2 \in [E_1|X'_1]$  are such that  $\pi(S_1) = \pi(S_2) = T$ . Then there is  $\varphi \in \mathcal{I}_1^{<\infty}$  that makes the

following diagram commute:

$$\begin{array}{ccc} S_1 & \overset{\varphi}{\dashrightarrow} & S_2 \\ \downarrow \pi & & \downarrow \pi \\ T & \xrightarrow{\text{id}_T} & T. \end{array}$$

Since  $B_1$  is  $\varphi$ -invariant, the sets

$$\{f \in [0; 1]^T : f \circ (\pi|_{S_1}) \in (B_1)_{S_1}\} \text{ and } \{f \in [0; 1]^T : f \circ (\pi|_{S_2}) \in (B_1)_{S_2}\}$$

coincide.

The next step is to show that the set  $B_2 := \bigcup_{T \in [E_2]^{<\infty}} (B_2)_T$  is Borel. For this, we observe that  $B_2$  is both analytic and co-analytic, since for  $f \in [E_2|X'_2]_{[0;1]}^{<\infty}$ ,

$$\begin{aligned} f \in B_2 &\iff \exists S \in [E_1|X'_1]^{<\infty} (\pi(S) = \text{dom}(f) \text{ and } f \circ (\pi|_S) \in B_1) \\ &\iff \forall S \in [E_1|X'_1]^{<\infty} (\pi(S) = \text{dom}(f) \implies f \circ (\pi|_S) \in B_1). \end{aligned}$$

Now we verify that  $B_2$  is  $\mathcal{I}_2$ -invariant  $\mu_2$ -a.e.; indeed it is  $\mathcal{I}_2$ -invariant on  $X'_2$ . Suppose that  $T_1, T_2 \in [E_2|X'_2]^{<\infty}$  and  $\varphi_2 \in \mathcal{I}_2^{<\infty}$  is a bijection between  $T_1$  and  $T_2$ . Choose any  $S_1, S_2 \in [E_1|X'_1]^{<\infty}$  such that  $\pi(S_1) = T_1$  and  $\pi(S_2) = T_2$ . There is  $\varphi_1 \in \mathcal{I}_1^{<\infty}$  that makes the following diagram commute:

$$\begin{array}{ccc} S_1 & \overset{\varphi_1}{\dashrightarrow} & S_2 \\ \downarrow \pi & & \downarrow \pi \\ T_1 & \xrightarrow{\varphi_2} & T_2. \end{array}$$

Since  $B_1$  is  $\varphi_1$ -invariant,  $B_2$  is  $\varphi_2$ -invariant.

Let  $p: \text{dom}(B_1) \rightarrow [0; 1]$  witness the correctness of  $B_1$ . Let  $\sigma: X'_2 \rightarrow X'_1$  be a function such that  $\sigma \circ \pi = \text{id}_{X'_2}$  and  $x E_2 y \iff \sigma(x) E_1 \sigma(y)$  (it is easy to construct such  $\sigma$  using the Axiom of Choice). For  $T \in \text{dom}(B_1)$ , let  $q(T) := p(\sigma(T))$ . If we denote  $S := \sigma(T)$ , then

$$\frac{q(T)}{1 - q(T)} \prod_{\substack{T' \in \text{dom}(B_2): \\ T' \cap T \neq \emptyset}} (1 - q(T')) = \frac{p(S)}{1 - p(S)} \prod_{\substack{S' \in \text{dom}(B_1): \\ S' \cap S \neq \emptyset}} (1 - p(S')) \geq \lambda^S((B_1)_S) = \lambda^T((B_2)_T).$$

Thus,  $B_2$  is correct.

Finally, if  $f$  is a  $\mu_2$ -solution for  $B_2$ , then  $f \circ \pi$  is a  $\mu_1$ -solution for  $B_1$ , since for any  $S \in \text{dom}(B_1) \cap [\text{dom}(f \circ \pi)]^{<\infty}$ , we have  $\pi(S) \in \text{dom}(B_2) \cap [\text{dom}(f)]^{<\infty}$ , so  $f|_{\pi(S)} \notin B_2$ , which by definition happens if and only if  $f \circ (\pi|_S) = (f \circ \pi)|_S \notin B_1$ .  $\blacksquare$

For a class  $\mathcal{C}$  of L-systems, define the class  $\mathcal{C}^*$  by

$$\mathbf{L} \in \mathcal{C}^* :\iff \exists \mathbf{L}' \in \mathcal{C} (\mathbf{L} \text{ admits a factor map to } \mathbf{L}').$$

Note that  $\mathcal{C}^* \supseteq \mathcal{C}$  and  $(\mathcal{C}^*)^* = \mathcal{C}^*$ . Suppose that  $\mathcal{C}$  is a class of L-systems satisfying the following two conditions:

(B1) if  $\mathbf{L} \in \mathcal{C}$ , then  $\text{HF}_0(\mathbf{L}) \in \mathcal{C}^*$ ;



(B2) if  $\mathbf{L} \in \mathcal{C}$  and  $B$  is a correct Borel instance over  $\mathbf{L}$ , then there exists a  $\mu_{\mathbf{L}}$ -solution  $f$  for  $B$  such that  $\mathbf{L}[f] \in \mathcal{C}^*$ .

Due to Proposition 5.6 and Lemma 5.7, if  $\mathcal{C}$  satisfies the above conditions, then  $\mathcal{C}^*$  has Properties (A1) and (A2) from §5.2.1.

### 5.2.3 Countable Borel groupoids and their actions

The following definition is an important ingredient of our argument.

**Definition 5.8** (Countable Borel groupoids). A *countable Borel groupoid*  $(R, \Gamma)$  is a structure consisting of a standard Borel space  $R$  together with a countable set  $\Gamma$  and Borel maps

$$\begin{aligned} \mathbf{a}: \Gamma \times R &\rightarrow R &: (\gamma, r) &\mapsto \gamma \cdot r && (\text{action}); \\ \mathbf{c}: \Gamma^2 \times R &\rightarrow \Gamma &: (\gamma_1, \gamma_2, r) &\mapsto \gamma_1 \circ_r \gamma_2 && (\text{composition}); \\ \mathbf{id}: R &\rightarrow \Gamma &: r &\mapsto \mathbf{1}_r && (\text{identity}); \\ \text{and } \mathbf{inv}: \Gamma \times R &\rightarrow \Gamma &: (\gamma, r) &\mapsto \gamma_r^{-1} && (\text{inverse}), \end{aligned}$$

satisfying the following axioms:

1. *consistency*: for all  $\gamma_1, \gamma_2 \in \Gamma$  and  $r \in R$ ,  $\gamma_1 \cdot (\gamma_2 \cdot r) = (\gamma_1 \circ_r \gamma_2) \cdot r$ ;
2. *associativity*: for all  $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$  and  $r \in R$ ,  $\gamma_1 \circ_r (\gamma_2 \circ_r \gamma_3) = (\gamma_1 \circ_{\gamma_3 \cdot r} \gamma_2) \circ_r \gamma_3$ ;
3. *identity*: for each  $r \in R$ ,  $\mathbf{1}_r \cdot r = r$  and for all  $\gamma \in \Gamma$ ,  $\mathbf{1}_{\gamma \cdot r} \circ_r \gamma = \gamma \circ_r \mathbf{1}_r = \gamma$ ;
4. *inverse*: For all  $r \in R$  and  $\gamma \in \Gamma$ ,  $\gamma_r^{-1} \circ_r \gamma = \mathbf{1}_r$  and  $\gamma \circ_{\gamma \cdot r} \gamma_r^{-1} = \mathbf{1}_{\gamma \cdot r}$ .

A countable group  $\Gamma$  can be canonically viewed as a countable Borel groupoid in the following way. Let  $R := \{r\}$  consist of a single point. For each  $\gamma \in \Gamma$ , let  $\gamma \cdot r := r$ . Now we just transfer compositions, the identity, and inverses directly from the group:

$$\begin{aligned} \gamma_1 \circ_r \gamma_2 &:= \gamma_1 \gamma_2; \\ \mathbf{1}_r &:= \mathbf{1}; \\ \gamma_r^{-1} &:= \gamma^{-1}. \end{aligned} \quad (5.1)$$

(Here  $\mathbf{1}$  denotes the identity element of  $\Gamma$ .)

A more general class of examples is given by Borel actions of countable groups.

Namely, let  $a: \Gamma \curvearrowright R$  be a Borel action of a countable group  $\Gamma$  on a standard Borel space  $R$ . Then  $(R, \Gamma)$  can be endowed with the structure of a countable Borel groupoid as follows: Set  $\gamma \cdot r := \gamma \cdot_a r$  for all  $\gamma \in \Gamma$ ,  $r \in R$ , and define compositions, identities, and inverses via (5.1) (i.e., they do not depend on  $r \in R$ ).

An interesting example of a countable Borel groupoid is produced by “bundling” all countable groups into a single algebraic structure. Let  $\mathcal{G}$  be the standard Borel space of all countably infinite groups with ground set  $\mathbb{N}$  (it can be viewed as a Borel subset of  $\{0, 1\}^{\mathbb{N}^3}$ ).

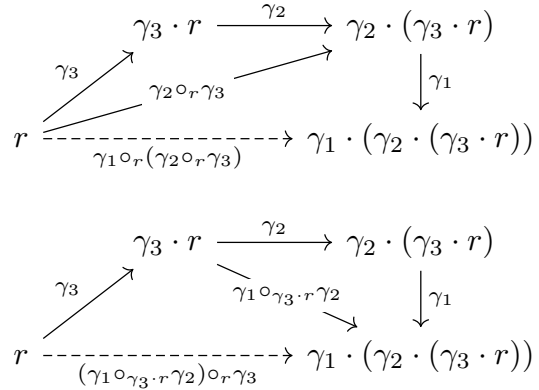


Figure 2: Associativity: the dashed arrows must coincide.

We define a countable Borel groupoid  $(\mathcal{G}, \mathbb{N})$  as follows. For each  $n \in \mathbb{N}$  and  $\Gamma \in \mathcal{G}$ , let  $n \cdot \Gamma := \Gamma$ . Now set  $n \circ_\Gamma m$  to be the product of  $n$  and  $m$  in  $\Gamma$ ;  $\mathbf{1}_\Gamma$  to be the identity element of  $\Gamma$ ; and  $n_\Gamma^{-1}$  to be the inverse of  $n$  in  $\Gamma$ .

The following proposition is a useful and easy-to-check condition that guarantees that a certain structure is a countable Borel groupoid.

**Proposition 5.9.** *Let  $R$  be a standard Borel space and let  $E$  be a countable Borel equivalence relation on  $R$ . Let  $\Gamma$  be a countable set and let  $\mathbf{a}: \Gamma \times R \rightarrow R: (\gamma, r) \mapsto \gamma \cdot r$  be a Borel function. Suppose that for each  $r \in R$ , the map  $\gamma \mapsto \gamma \cdot r$  is a bijection between  $\Gamma$  and  $[r]_E$ . Then there is a unique countable Borel groupoid structure on  $(R, \Gamma)$  with  $\mathbf{a}$  as its action map.*

*Proof.* For  $r_1, r_2 \in R$  with  $r_1 E r_2$ , let  $\delta(r_1, r_2)$  be the unique element of  $\Gamma$  such that  $r_2 = \delta(r_1, r_2) \cdot r_1$ . The only consistent way to turn  $(R, \Gamma)$  into a countable Borel groupoid is as follows:

$$\begin{aligned} \gamma_1 \circ_r \gamma_2 &:= \delta(r, \gamma_1 \cdot (\gamma_2 \cdot r)); \\ \mathbf{1}_r &:= \delta(r, r); \\ \gamma_r^{-1} &:= \delta(\gamma \cdot r, r). \end{aligned}$$

A straightforward verification shows that all the axioms are satisfied. ■

Now we proceed to the definition of Borel actions of countable Borel groupoids.

**Definition 5.10** (Actions). Let  $(R, \Gamma)$  be a countable Borel groupoid. A (Borel) *action*  $(\rho, a)$  of  $(R, \Gamma)$  on a standard Borel space  $X$  is a pair of Borel maps  $\rho: X \rightarrow R$ ,  $a: \Gamma \times X \rightarrow X: (\gamma, x) \mapsto \gamma \cdot_a x$  satisfying the following conditions:

1. *equivariance*: for all  $x \in X$  and  $\gamma \in \Gamma$ ,  $\rho(\gamma \cdot_a x) = \gamma \cdot \rho(x)$ ;
2. *identity*: for all  $x \in X$ ,  $\mathbf{1}_{\rho(x)} \cdot_a x = x$ ;
3. *compatibility*: for all  $x \in X$  and  $\gamma_1, \gamma_2 \in \Gamma$ ,  $\gamma_1 \cdot_a (\gamma_2 \cdot_a x) = (\gamma_1 \circ_{\rho(x)} \gamma_2) \cdot_a x$ .

As with group actions, we will usually simply write  $\gamma \cdot x$  for  $\gamma \cdot_a x$ .

Clearly, a (left) group action  $\Gamma \curvearrowright X$  is also a countable Borel groupoid action if  $\Gamma$  is understood as a countable Borel groupoid. Now suppose that a countable group  $\Gamma$  acts (in a Borel way) on a standard Borel space  $R$  and let  $(R, \Gamma)$  be the corresponding countable Borel groupoid. Let  $(\rho, a)$  be an action of  $(R, \Gamma)$  on  $X$ . Then the identity and compatibility conditions from Definition 5.10 imply that  $a$  is an action of  $\Gamma$  on  $X$ , while the equivariance condition stipulates that the map  $\rho: X \rightarrow R$  is  $\Gamma$ -equivariant. In other words, Borel actions of  $(R, \Gamma)$  exactly correspond to  $\Gamma$ -spaces equipped with a Borel  $\Gamma$ -equivariant map to  $R$ .

If  $(\mathcal{G}, \mathbb{N})$  is the countable Borel groupoid of all countable groups, then an action of  $(\mathcal{G}, \mathbb{N})$  on  $X$  consists of a Borel map  $\rho: X \rightarrow \mathcal{G}$  and a  $\Gamma$ -action on  $\rho^{-1}(\Gamma)$  for each  $\Gamma \in \mathcal{G}$ .

**Definition 5.11** (Shift actions). Let  $(R, \Gamma)$  be a countable Borel groupoid and let  $Y$  be a standard Borel space. The *Y-shift action* of  $(R, \Gamma)$  is the action  $(\rho, a)$  of  $(R, \Gamma)$  on the space  $R \times Y^\Gamma$ , defined as follows. For each  $(r, \vartheta) \in R \times Y^\Gamma$ , let  $\rho(r, \vartheta) := r$  and for  $\gamma \in \Gamma$ , define

$$\gamma \cdot_a (r, \vartheta) := (\gamma \cdot r, \vartheta'),$$

where  $\vartheta' \in Y^\Gamma$  is given by

$$\vartheta'(\delta) := \vartheta(\delta \circ_r \gamma)$$

for all  $\delta \in \Gamma$ .

A straightforward verification shows that the  $Y$ -shift action is indeed an action of  $(R, \Gamma)$ . We demonstrate the proof here to help the reader get familiar with the definitions. The equivariance condition is satisfied trivially. For the identity condition, observe that if  $x = (r, \vartheta) \in R \times Y^\Gamma$ , then

$$\mathbf{1}_{\rho(x)} \cdot x = \mathbf{1}_r \cdot (r, \vartheta) = (\mathbf{1}_r \cdot r, \vartheta') = (r, \vartheta'),$$

where for each  $\delta \in \Gamma$ ,

$$\vartheta'(\delta) = \vartheta(\delta \circ_r \mathbf{1}_r) = \vartheta(\delta),$$

so  $\vartheta' = \vartheta$ , as desired. Finally, for the compatibility condition, we have

$$\gamma_1 \cdot (\gamma_2 \cdot x) = \gamma_1 \cdot (\gamma_2 \cdot (r, \vartheta)) = \gamma_1 \cdot (\gamma_2 \cdot r, \vartheta') = (\gamma_1 \cdot (\gamma_2 \cdot r), \vartheta'') = ((\gamma_1 \circ_r \gamma_2) \cdot r, \vartheta''),$$

where for  $\delta \in \Gamma$ ,

$$\vartheta''(\delta) = \vartheta'(\delta \circ_{\gamma_2 \cdot r} \gamma_1) = \vartheta((\delta \circ_{\gamma_2 \cdot r} \gamma_1) \circ_r \gamma_2) = \vartheta(\delta \circ_r (\gamma_1 \circ_r \gamma_2)),$$

so

$$\gamma_1 \cdot (\gamma_2 \cdot x) = (\gamma_1 \circ_r \gamma_2) \cdot x,$$

as desired. Note that for a countable group  $\Gamma$ , Definition 5.11 is equivalent to the usual definition of the (left) shift action of  $\Gamma$  on  $Y^\Gamma$ .

By analogy with group actions, we can define L-systems corresponding to actions of countable Borel groupoids. Namely, let  $(\rho, a): (R, \Gamma) \curvearrowright X$  be a Borel action of a countable Borel groupoid  $(R, \Gamma)$  on a standard Borel space  $X$ . Let  $E_a$  be the *orbit equivalence relation*, defined by

$$xE_a y : \Longleftrightarrow \exists \gamma \in \Gamma (\gamma \cdot x = y).$$

The relation  $E_a$  is easily seen to be a countable Borel equivalence relation. Note that it does not depend on  $\rho$ . Let  $\mathcal{I}_{(\rho, a)}$  denote the isomorphism structure on  $E_a$  such that a bijection  $\varphi: C_1 \rightarrow C_2$  belongs to  $\mathcal{I}_{(\rho, a)}$  if and only if  $C_1$  and  $C_2$  are  $E_a$ -classes and  $\varphi$  is  $(R, \Gamma)$ -equivariant, i.e., for all  $x \in C_1$  and  $\gamma \in \Gamma$ ,  $\rho(\varphi(x)) = \rho(x)$  and  $\gamma \cdot \varphi(x) = \varphi(\gamma \cdot x)$ . If  $\mu \in \mathbf{P}(X)$ , then let  $\mathbf{L}(\rho, a, \mu)$  denote the L-system  $(X, E_a, \mathcal{I}_{(\rho, a)}, \mu)$ . In the case when  $|R| = 1$ , i.e.,  $(R, \Gamma)$  is a group, this definition coincides with the one given previously for group actions.

We will be mostly concerned with the properties of L-systems induced by shift actions of countable Borel groupoids. More precisely:

**Definition 5.12** (Lebesgue L-systems). A *Lebesgue L-system* is any L-system of the form  $\mathbf{L}(\rho, a, \mu \times \nu^\Gamma)$ , where  $(\rho, a): (R, \Gamma) \curvearrowright R \times Y^\Gamma$  is the  $Y$ -shift action of a countable Borel groupoid  $(R, \Gamma)$  for some standard Borel space  $Y$ ,  $\mu \in \mathbf{P}(R)$ , and  $\nu \in \mathbf{P}(Y)$  is atomless.

Due to the measure isomorphism theorem, it is enough to consider Lebesgue L-systems induced by the  $[0; 1]$ -shift action of  $(R, \Gamma)$  with  $\nu = \lambda$ . However, sometimes it will be more convenient to consider other choices for  $Y$  and  $\nu$ ; in particular, we will often assume that  $Y = [0; 1]^S$  and  $\nu = \lambda^S$  for some countable set  $S$ .

### 5.2.4 Factors of L-systems induced by actions of countable Borel groupoids

Let  $(R, \Gamma)$  be a countable Borel groupoid and let  $(\rho, a): (R, \Gamma) \curvearrowright X$  be a Borel action of  $(R, \Gamma)$  on a standard Borel space  $X$ . The action  $(\rho, a)$  is said to be *free* if for all  $x \in X$  and  $\gamma \in \Gamma$ ,  $(\gamma \cdot x = x) \iff (\gamma = \mathbf{1}_{\rho(x)})$ . The *free part* of  $(\rho, a)$  (notation:  $\text{Free}(\rho, a)$  or  $\text{Free}(X)$  if the action is clear from the context) is the largest  $E_a$ -invariant subset  $X' \subseteq X$  on which the action is free. For  $\mu \in \mathcal{P}(X)$ , an action is *free  $\mu$ -a.e.* if its free part is  $\mu$ -conull. Note that the free part of an action  $(R, \Gamma) \curvearrowright X$  is always an invariant Borel subset of  $X$ . Also note that if  $x \in \text{Free}(X)$ , then the map  $\gamma \mapsto \gamma \cdot x$  is a bijection between  $\Gamma$  and the orbit of  $x$ .

**Proposition 5.13.** *Let  $(R, \Gamma)$  be a countable Borel groupoid and let  $\mu \in \mathcal{P}(R)$ . Let  $Y$  be a standard Borel space and let  $\nu \in \mathcal{P}(Y)$  be atomless. Then the  $Y$ -shift action of  $(R, \Gamma)$  on  $R \times Y^\Gamma$  is free  $(\mu \times \nu^\Gamma)$ -a.e.*

*Proof.* It is enough to notice that  $\text{Free}(R \times Y^\Gamma) \supseteq R \times F$ , where

$$F := \{\vartheta \in Y^\Gamma : \forall \gamma, \delta \in \Gamma (\vartheta(\gamma) = \vartheta(\delta) \iff \gamma = \delta)\},$$

and  $F$  is  $\nu^\Gamma$ -conull. ■

The following lemma will be useful in verifying that certain maps between L-systems induced by actions of countable Borel groupoids are factor maps.

**Lemma 5.14.** *Let  $(R, \Gamma)$  be a countable Borel groupoid and let  $(\rho_1, a_1): (R, \Gamma) \curvearrowright X_1$  and  $(\rho_2, a_2): (R, \Gamma) \curvearrowright X_2$  be Borel actions of  $(R, \Gamma)$  on standard Borel spaces  $X_1$  and  $X_2$ . Let  $\mu_1 \in \mathcal{P}(X_1)$  and  $\mu_2 \in \mathcal{P}(X_2)$ . Suppose that  $(\rho_2, a_2)$  is  $\mu_2$ -a.e. free. Let  $\pi: X'_1 \rightarrow X_2$  be a measure-preserving  $(R, \Gamma)$ -equivariant Borel map defined on an  $E_{a_1}$ -invariant  $\mu_1$ -conull Borel subset  $X'_1 \subseteq X_1$ . Then  $\pi$ , possibly restricted to a smaller invariant conull Borel subset of  $X_1$ , is a factor map from  $\mathbf{L}(\rho_1, a_1, \mu_1)$  to  $\mathbf{L}(\rho_2, a_2, \mu_2)$ .*

*Proof.* For  $k \in \{1, 2\}$ , let

$$E_k := E_{a_k} \quad \text{and} \quad \mathcal{I}_k := \mathcal{I}_{(\rho_k, a_k)}.$$

Let  $C \subseteq X'_1$  be an  $E_1$ -class. The equivariance of  $\pi$  implies that  $\pi(C)$  is an  $E_2$ -class. Since  $(\rho_2, a_2)$  is free  $\mu_2$ -a.e., we may assume that  $\pi(C) \subseteq \text{Free}(X_2)$ . Then the map  $\pi|_C: C \rightarrow \pi(C)$  is a bijection.

It remains to check the existence of  $\varphi_1 \in \mathcal{I}_1^{<\infty}$  that closes the following diagram:

$$\begin{array}{ccc} S_1 & \overset{\varphi_1}{\dashrightarrow} & S_2 \\ \downarrow \pi & & \downarrow \pi \\ \pi(S_1) & \xrightarrow{\varphi_2} & \pi(S_2). \end{array}$$

Let  $\overline{\varphi}_2: [\pi(S_1)]_{E_2} \rightarrow [\pi(S_2)]_{E_2}$  be an equivariant bijection that extends  $\varphi_2$ . Again, since  $(\rho_2, a_2)$  is free  $\mu_2$ -a.e., we can assume that the maps

$$\pi|_{[S_1]_{E_1}}: [S_1]_{E_1} \rightarrow [\pi(S_1)]_{E_2},$$

$$\pi|_{[S_2]_{E_1}}: [S_2]_{E_1} \rightarrow [\pi(S_2)]_{E_2}$$

are bijections. Hence,  $\bar{\varphi}_1 := (\pi|_{[S_2]_{E_1}})^{-1} \circ \bar{\varphi}_2 \circ (\pi|_{[S_1]_{E_1}})$  is an equivariant bijection from  $[S_1]_{E_1}$  to  $[S_2]_{E_1}$ ,  $\bar{\varphi}_1(S_1) = S_2$ , and the following diagram commutes:

$$\begin{array}{ccc} [S_1]_{E_1} & \xrightarrow{\bar{\varphi}_1} & [S_2]_{E_1} \\ \downarrow \pi & & \downarrow \pi \\ [\pi(S_1)]_{E_2} & \xrightarrow{\bar{\varphi}_2} & [\pi(S_2)]_{E_2}. \end{array}$$

Thus, we can set  $\varphi_1 := \bar{\varphi}_1|_{S_1}$ . ■

Let  $\Gamma$  be a countable group and let  $a_1: \Gamma \curvearrowright (X_1, \mu_1)$ ,  $a_2: \Gamma \curvearrowright (X_2, \mu_2)$  be two measure-preserving actions of  $\Gamma$ . If the action on  $X_2$  is free  $\mu_2$ -a.e., then Lemma 5.14 implies that a factor map  $\pi: (X_1, \mu_1) \rightarrow (X_2, \mu_2)$  in the sense of ergodic theory induces a factor map between the L-systems  $\mathbf{L}(a_1, \mu_1)$  and  $\mathbf{L}(a_2, \mu_2)$ .

### 5.2.5 Closure properties of the class of Lebesgue L-systems

In this paragraph we show that the class of Lebesgue L-systems is closed under (some) expansions and under amplifications. We start with the following simple observation.

**Lemma 5.15.** *Let  $(R, \Gamma)$  be a countable Borel groupoid. Let  $\mathbf{L} = \mathbf{L}(\rho, a, \mu \times (\lambda^2)^\Gamma)$ , where  $\mu \in \mathcal{P}(R)$ , be the Lebesgue L-system produced by the  $[0; 1]^2$ -shift action of  $(R, \Gamma)$ . Let  $Y$  be a standard Borel space and let  $f: R \times ([0; 1]^2)^\Gamma \rightarrow Y$  be a Borel function that does not depend on the third coordinate, i.e., for all  $r \in R$ ,  $\vartheta, \omega, \omega' \in [0; 1]^\Gamma$ ,*

$$f(r, \vartheta, \omega) = f(r, \vartheta, \omega').$$

*Then  $\mathbf{L}[f]$  admits a factor map to a Lebesgue L-system.*

*Proof.* Let  $Q := R \times [0; 1]^\Gamma$ . The  $[0; 1]$ -shift action of  $(R, \Gamma)$  on  $Q$  turns  $(Q, \Gamma)$  into a countable Borel groupoid via

$$\begin{aligned} \gamma_1 \circ_{(r, \vartheta)} \gamma_2 &:= \gamma_1 \circ_r \gamma_2; \\ \mathbf{1}_{(r, \vartheta)} &:= \mathbf{1}_r; \\ \gamma_{(r, \vartheta)}^{-1} &:= \gamma_r^{-1}. \end{aligned}$$

Let  $(\sigma, a')$  denote the  $[0; 1]$ -shift action of  $(Q, \Gamma)$ . If we identify  $([0; 1]^2)^\Gamma$  with  $[0; 1]^\Gamma \times [0; 1]^\Gamma$  in the usual way, then  $R \times [0; 1]^\Gamma \times [0; 1]^\Gamma = Q \times [0; 1]^\Gamma$  and, in fact,  $a = a'$ . By definition, we have that for all  $r \in R$ ,  $\vartheta, \omega \in [0; 1]^\Gamma$ ,

$$\sigma(r, \vartheta, \omega) = (r, \vartheta),$$

so the value of  $f(x)$  is determined by  $\sigma(x)$  for all  $x$ .

Let  $\mathbf{L}' := \mathbf{L}(\sigma, a, \mu \times (\lambda^2)^\Gamma)$  be the Lebesgue L-system induced by  $(\sigma, a)$ . We claim that the identity function  $\text{id}_{R \times ([0; 1]^2)^\Gamma}$  is a factor map from  $\mathbf{L}[f]$  to  $\mathbf{L}'$ . The only thing we need to check is Condition 3. of Definition 5.5. We claim that, in fact,  $\mathcal{I}_{(\sigma, a)} \subseteq \mathcal{I}_{(\rho, a)}[f]$ . Indeed, if a bijection  $\varphi: C_1 \rightarrow C_2$  between  $E_a$ -classes  $C_1$  and  $C_2$  belongs to  $\mathcal{I}_{(\sigma, a)}$ , then for all  $x \in C_1$ ,  $\sigma(\varphi(x)) = \sigma(x)$ , so  $\rho(\varphi(x)) = \rho(x)$  and  $f(\varphi(x)) = f(x)$ . Thus,  $\varphi \in \mathcal{I}_{(\rho, a)}[f]$ , as desired. ■

The proof of the next lemma is a bit more involved than that of Lemma 5.15. Recall that for  $\varphi: A \rightarrow B$ ,  $\tilde{\varphi}: \text{HF}_0(A) \rightarrow \text{HF}_0(B)$  denotes the amplification of  $\varphi$ ; and for an equivalence relation  $E$  on  $A$ ,  $\tilde{E}$  denotes the amplification of  $E$ , which is an equivalence relation on  $\text{HF}_0(E)$ .

**Lemma 5.16.** *If  $\mathbf{L}$  is a Lebesgue  $L$ -system, then  $\text{HF}_0(\mathbf{L})$  factors to a Lebesgue  $L$ -system.*

*Proof.* Suppose that  $\mathbf{L}$  is induced by a shift action of a countable Borel groupoid  $(R, \Gamma)$ . We will proceed in three steps. First, we will construct a countable Borel groupoid  $(Q, \Delta)$ , where  $\Delta = \text{HF}_0(\Gamma)$ . Then we will show that  $\text{HF}_0(\mathbf{L})$  is induced by an (almost) free action of  $(Q, \Delta)$ . Finally, we will define a measure-preserving  $(Q, \Delta)$ -equivariant map from this action to the  $[0; 1]$ -shift action of  $(Q, \Delta)$ , which will give us a desired factor map, thanks to Lemma 5.14.

**Step 1.** Let  $\mu \in \text{P}(R)$  and consider the  $[0; 1]$ -shift action  $(R, \Gamma) \curvearrowright R \times [0; 1]^\Gamma$ . Denote the corresponding orbit equivalence relation by  $E$ . Define

$$Q := \text{HF}_0(E | \text{Free}(R \times [0; 1]^\Gamma)) \quad \text{and} \quad \Delta := \text{HF}_0(\Gamma).$$

Note that for each  $x \in \text{Free}(R \times [0; 1]^\Gamma)$ , the following map is a bijection between  $\Gamma$  and  $[x]_E$ :

$$\varphi_x: \Gamma \rightarrow [x]_E: \gamma \mapsto \gamma \cdot x.$$

Therefore, its amplification  $\tilde{\varphi}_x: \Delta \rightarrow \text{HF}_0([x]_E) = [x]_{\tilde{E}}$  is a bijection between  $\Delta$  and  $[x]_{\tilde{E}}$ .

Fix a Borel map  $x_0: Q \rightarrow \text{Free}(R \times [0; 1]^\Gamma)$  such that for all  $q \in Q$ ,  $x_0(q) \in \mathcal{U}(q)$ , and let  $\tilde{\varphi}_q := \tilde{\varphi}_{x_0(q)}$ . Then for all  $q \in Q$ ,  $\tilde{\varphi}_q$  is a bijection from  $\Delta$  to  $[x_0(q)]_{\tilde{E}} = [q]_{\tilde{E}}$ . For  $q \in Q$  and  $\delta \in \Delta$ , define

$$\delta \cdot q := \tilde{\varphi}_q(\delta).$$

Note that we have

$$\delta \cdot q = \tilde{\varphi}_q(\delta) = \tilde{\varphi}_{x_0(q)}(\delta) = \delta \cdot x_0(q).$$

Observe that  $\Gamma \subseteq \Delta$  and  $\text{Free}(R \times [0; 1]^\Gamma) \subseteq Q$ ; hence the value  $\gamma \cdot x$ , where  $x \in \text{Free}(R \times [0; 1]^\Gamma)$  and  $\gamma \in \Gamma$ , can be computed in two ways:  $\gamma$  can be viewed as an element of  $\Gamma$  acting on  $R \times [0; 1]^\Gamma$ , or it can be viewed as an element of  $\Delta$  acting on  $Q$ . However, it is easy to see that both interpretations produce the same outcome, so our notation is consistent.

Since  $\tilde{\varphi}_q: \Delta \rightarrow [q]_{\tilde{E}}$  is a bijection for each  $q \in Q$ , by Proposition 5.9,  $(Q, \Delta)$  is equipped with a unique countable Borel groupoid structure.

**Step 2.** Now we turn to the Lebesgue  $L$ -system  $\mathbf{L}$ . Suppose that  $\mathbf{L} = \mathbf{L}(\rho, a, \mu \times (\lambda \times \nu)^\Gamma)$ , where  $(\rho, a): (R, \Gamma) \curvearrowright R \times ([0; 1] \times Y)^\Gamma$  is the  $([0; 1] \times Y)$ -shift action of  $(R, \Gamma)$ ,  $\mu \in \text{P}(R)$ , and  $\nu \in \text{P}(Y)$  is atomless. Here  $Y$  is a standard Borel space; we will specify a concrete choice for  $Y$  later. Let

$$F := \text{Free}(R \times [0; 1]^\Gamma) \times Y^\Gamma.$$

Then  $F$  is a conull  $E_a$ -invariant Borel subset of  $\text{Free}(\rho, a)$ . We will now define a free action of  $(Q, \Delta)$  on  $H := \text{HF}_0(E_a | F)$  (which is a conull  $\tilde{E}_a$ -invariant Borel subset of  $\text{HF}_0(E_a)$ ).

The construction is analogous to the one from Step 1. For each  $x \in F$ , let  $\varphi_x: \Gamma \rightarrow [x]_{E_a}$  be given by

$$\varphi_x: \Gamma \rightarrow [x]_{E_a}: \gamma \mapsto \gamma \cdot x.$$

Then  $\varphi_x$  establishes a bijection between  $\Gamma$  and  $[x]_{E_a}$ . Therefore,  $\tilde{\varphi}_x: \Delta \rightarrow [x]_{\tilde{E}_a}$  is a bijection between  $\Delta$  and  $[x]_{\tilde{E}_a}$ . Let  $\sigma: F \rightarrow \text{Free}(R \times [0; 1]^\Gamma)$  denote the projection on the first two coordinates, i.e.,

$$\sigma(r, \vartheta, y) := (r, \vartheta)$$

for all  $(r, \vartheta) \in \text{Free}(R \times [0; 1]^\Gamma)$  and  $y \in Y^\Gamma$ . Note that for every  $h \in H$ ,  $\tilde{\sigma}(h) \in Q$ . Moreover, the map  $\sigma|_{\mathcal{U}(h)}: \mathcal{U}(h) \rightarrow \mathcal{U}(\tilde{\sigma}(h))$  is a bijection. Let  $x_0(h)$  be the unique element of  $\mathcal{U}(h)$  such that

$$\sigma(x_0(h)) = x_0(\tilde{\sigma}(h)).$$

For  $\delta \in \Delta$ , define  $\tilde{\varphi}_h := \tilde{\varphi}_{x_0(h)}$ . Then  $\tilde{\varphi}_h: \Delta \rightarrow [h]_{\tilde{E}_a}$  is a bijection. Hence, if we let

$$b: \Delta \times H: (\delta, h) \mapsto \delta \cdot h := \tilde{\varphi}_h(\delta),$$

then  $(\tilde{\sigma}, b)$  is a free action of  $(Q, \Delta)$  on  $H$ . Note that we again have

$$\delta \cdot h = \tilde{\varphi}_h(\delta) = \tilde{\varphi}_{x_0(h)}(\delta) = \delta \cdot x_0(h).$$

Similarly to Step 1, the value  $\gamma \cdot (x, y)$  for  $(x, y) \in F$  and  $\gamma \in \Gamma$  does not depend on whether we construe  $\gamma$  as an element of  $\Gamma$  (acting on  $R \times [0; 1]^\Gamma \times Y^\Gamma$ ) or as an element of  $\Delta$  (acting on  $H$ ). Moreover, it is clear that the restriction of  $\text{HF}_0(\mathbf{L})$  to  $H$  coincides with  $\mathbf{L}(\tilde{\sigma}, b, \mu \times (\lambda \times \nu)^\Gamma)$ .

**Step 3.** So far we have constructed a countable Borel groupoid  $(Q, \Delta)$  and a free action  $(\tilde{\sigma}, b): (Q, \Delta) \curvearrowright H$ , which essentially (i.e., up to an invariant null set) induces the L-system  $\text{HF}_0(\mathbf{L})$ . Now we will define a factor map from  $\text{HF}_0(\mathbf{L})$  to the L-system induced by the shift action of  $(Q, \Delta)$  on  $Q \times [0; 1]^\Delta$ .

For this, we choose  $Y$  to be  $[0; 1]^\Delta$  and  $\nu$  to be  $\lambda^\Delta$ . Consider any  $h \in H$ . Suppose that  $x_0(h) = (x, y)$ , where  $x \in \text{Free}(R \times [0; 1]^\Gamma)$  and  $y \in Y^\Gamma = ([0; 1]^\Delta)^\Gamma = [0; 1]^{\Gamma \times \Delta}$ . Define  $\beta(h) \in [0; 1]$  by

$$\beta(h) := y(\mathbf{1}_{\rho(x)}, \mathbf{1}_{\tilde{\sigma}(h)}).$$

Here  $\rho(x) \in R$ , so  $\mathbf{1}_{\rho(x)} \in \Gamma$ ; and  $\tilde{\sigma}(h) \in Q$ , so  $\mathbf{1}_{\tilde{\sigma}(h)} \in \Delta$ . Now define  $\beta^\Delta: H \rightarrow [0; 1]^\Delta$  by setting, for all  $h \in H$  and  $\delta \in \Delta$ ,

$$\beta^\Delta(h)(\delta) := \beta(\delta \cdot h).$$

By construction, the map

$$(\tilde{\sigma}, \beta^\Delta): H \rightarrow Q \times [0; 1]^\Delta$$

is  $(Q, \Delta)$ -equivariant. Due to Lemma 5.14, it only remains to check that this map is measure-preserving.

Since we have the freedom to choose the measure on  $Q$ , we can take it to be

$$\tilde{\sigma}_*(\mu \times \lambda^\Gamma \times \lambda^{\Gamma \times \Delta}),$$

so we only need to check that

$$\beta_*^\Delta(\mu \times \lambda^\Gamma \times \lambda^{\Gamma \times \Delta}) = \lambda^\Delta.$$

Since  $\mu \times \lambda^\Gamma \times \lambda^{\Gamma \times \Delta}$  is concentrated on  $F$ , it is enough to verify that

$$(\beta^\Delta|_F)_*(\mu \times \lambda^\Gamma \times \lambda^{\Gamma \times \Delta}) = \lambda^\Delta.$$

To this end, we will show that for each  $x \in \text{Free}(R \times [0; 1]^\Gamma)$ , the map

$$\beta_x^\Delta : [0; 1]^{\Gamma \times \Delta} \rightarrow [0; 1]^\Delta : y \mapsto \beta^\Delta(x, y)$$

satisfies  $(\beta_x^\Delta)_*(\lambda^{\Gamma \times \Delta}) = \lambda^\Delta$ ; an application of Fubini's theorem will complete the proof.

Fix some  $x \in \text{Free}(R \times [0; 1]^\Gamma)$ . For each  $\delta \in \Delta$ , let  $\gamma_{x,\delta}$  be the unique element of  $\Gamma$  such that

$$x_0(\delta \cdot x) = \gamma_{x,\delta} \cdot x.$$

Observe that the map

$$\Delta \rightarrow \Gamma \times \Delta : \delta \mapsto (\gamma_{x,\delta}, \mathbf{1}_{\delta \cdot x})$$

is injective. Indeed, we have

$$\tilde{\varphi}_x(\delta) = \delta \cdot x = \mathbf{1}_{\delta \cdot x} \cdot (\delta \cdot x) = \mathbf{1}_{\delta \cdot x} \cdot x_0(\delta \cdot x) = \mathbf{1}_{\delta \cdot x} \cdot (\gamma_{x,\delta} \cdot x),$$

and the map  $\tilde{\varphi}_x$  is injective. Now let  $y \in [0; 1]^{\Gamma \times \Delta}$ . We have

$$x_0(\delta \cdot (x, y)) = \gamma_{x,\delta} \cdot (x, y) = (\gamma_{x,\delta} \cdot x, y'),$$

and

$$\tilde{\sigma}(\delta \cdot (x, y)) = \delta \cdot x,$$

where  $y'$  is a particular element of  $[0; 1]^{\Gamma \times \Delta}$ . Therefore,

$$\beta_x^\Delta(y)(\delta) = \beta(\delta \cdot (x, y)) = y'(\mathbf{1}_{\rho(\gamma_{x,\delta} \cdot x)}, \mathbf{1}_{\delta \cdot x}) = y'(\mathbf{1}_{\gamma_{x,\delta} \cdot \rho(x)}, \mathbf{1}_{\delta \cdot x}).$$

Since

$$\mathbf{1}_{\gamma_{x,\delta} \cdot \rho(x)} \circ \rho(x) \gamma_{x,\delta} = \gamma_{x,\delta},$$

by the definition of the shift action, we get

$$y'(\mathbf{1}_{\gamma_{x,\delta} \cdot \rho(x)}, \mathbf{1}_{\delta \cdot x}) = y(\gamma_{x,\delta}, \mathbf{1}_{\delta \cdot x}).$$

To summarize,

$$\beta_x^\Delta(y)(\delta) = y(\gamma_{x,\delta}, \mathbf{1}_{\delta \cdot x}).$$

In other words,  $\beta_x^\Delta$  acts as the projection on the set of coordinates  $\{(\gamma_{x,\delta}, \mathbf{1}_{\delta \cdot x}) : \delta \in \Delta\}$ . Therefore, it pushes  $\lambda^{\Gamma \times \Delta}$  forward to  $\lambda^\Delta$ , as desired.  $\blacksquare$

### 5.2.6 The Moser–Tardos algorithm for Lebesgue L-systems

In this paragraph we use Moser–Tardos theory to show that any correct Borel instance over a Lebesgue L-system  $\mathbf{L}$  admits a  $\mu_{\mathbf{L}}$ -solution. To do this, we will reduce  $B$  to a family  $(B_r)_{r \in R}$  of correct instances over  $\Gamma$  indexed by the elements of  $R$ , where  $(R, \Gamma)$  is the countable Borel groupoid whose shift action induces  $\mathbf{L}$ .

**Lemma 5.17.** *Let  $\mathbf{L}$  be a Lebesgue L-system. Then every correct Borel instance over  $\mathbf{L}$  has a  $\mu_{\mathbf{L}}$ -solution.*



*Proof.* Let  $(R, \Gamma)$  be a countable Borel groupoid, let  $\mu \in \mathcal{P}(R)$ , and let  $(\rho, a): (R, \Gamma) \curvearrowright R \times [0; 1]^{\Gamma \times \mathbb{N}}$  be the  $[0; 1]^{\Gamma \times \mathbb{N}}$ -shift action of  $(R, \Gamma)$ . Let  $\mathbf{L} := \mathbf{L}(\rho, a, \mu \times \lambda^{\Gamma \times \mathbb{N}})$ . We use the following notation:

$$\begin{aligned} X &:= R \times [0; 1]^{\Gamma \times \mathbb{N}}; \\ E &:= E_a; \\ \mathcal{I} &:= \mathcal{I}_{(\rho, a)}. \end{aligned}$$

Suppose  $B$  is a correct Borel instance over  $\mathbf{L}$ . Due to Propositions 2.8 and 2.10, it is enough to show that there exists a Borel table  $\beta: X \rightarrow [0; 1]^{\mathbb{N}}$  such that

$$(\mu \times \lambda^{\Gamma \times \mathbb{N}})(\{x \in X : \forall \gamma \in \Gamma (\gamma \cdot x \text{ is } \beta\text{-stable})\}) = 1. \quad (5.2)$$

We claim that the map

$$\beta: X \rightarrow [0; 1]^{\mathbb{N}}: (r, \vartheta) \mapsto \vartheta(\mathbf{1}_r)$$

satisfies (5.2). Note that for every  $\gamma \in \Gamma$ ,

$$\beta(\gamma \cdot (r, \vartheta)) = \vartheta(\gamma).$$

For each  $x \in X$ , there is a surjection

$$\varphi_x: \Gamma \rightarrow [x]_E: \gamma \mapsto \gamma \cdot x$$

from  $\Gamma$  onto  $[x]_E$ . Since the action  $(\rho, a)$  is free a.e.,  $\varphi_x$  is a bijection for almost all  $x \in X$ . Hence, for almost all  $x \in X$ ,  $\varphi_x$  can be used to create a “pullback”  $B_x$  of  $B \cap [[x]_E]_{[0; 1]}^{<\infty}$ , which is a correct instance over  $\Gamma$ . Formally, for  $S \in [\Gamma]^{<\infty}$  and  $f: S \rightarrow [0; 1]$ , we set

$$f \in (B_x)_S : \iff f \circ (\varphi_x|_S) \in B_{\varphi_x(S)}.$$

Note that whenever  $r \in R$ ,  $\vartheta, \omega \in [0; 1]^{\Gamma \times \mathbb{N}}$ , and both  $(r, \vartheta)$  and  $(r, \omega)$  belong to the free part of  $(\rho, a)$ , the map

$$\gamma \cdot (r, \vartheta) \mapsto \gamma \cdot (r, \omega)$$

is a well-defined  $(R, \Gamma)$ -equivariant bijection between  $[(r, \vartheta)]_E$  and  $[(r, \omega)]_E$ . Therefore, since  $B$  is  $\mathcal{I}$ -invariant a.e., the following definition makes sense for almost all  $r \in R$ :

$$B_r := B_{(r, \vartheta)} \text{ for almost all } \vartheta \in [0; 1]^{\Gamma \times \mathbb{N}}.$$

Now, for almost every  $r \in R$  and for all  $\gamma \in \Gamma$ , using Corollary 2.7, we obtain

$$\begin{aligned} &\lambda^{\Gamma \times \mathbb{N}}(\{\vartheta \in [0; 1]^{\Gamma \times \mathbb{N}} : \gamma \cdot (r, \vartheta) \text{ is } \beta\text{-stable with respect to } B\}) \\ &= \lambda^{\Gamma \times \mathbb{N}}(\{\vartheta \in [0; 1]^{\Gamma \times \mathbb{N}} : \gamma \text{ is } \vartheta\text{-stable with respect to } B_r\}) = 1. \end{aligned}$$

An application of Fubini’s theorem yields (5.2). ■

### 5.2.7 Completing the proof of Theorem 5.4

Now we have all the necessary ingredients to prove the following generalization of Theorem 5.4:

**Theorem 5.18** (The measurable LLL for Lebesgue L-systems). *Let  $\mathbf{L}$  be an L-system that admits a factor map to a Lebesgue L-system. Then  $\mathbf{L}$  satisfies the LLL.*

*Proof.* We need to verify that the class  $\mathcal{C}$  of Lebesgue L-systems satisfies Conditions (B1) and (B2) from §5.2.2. Condition (B1) is given by Lemma 5.16. It remains to show that if  $\mathbf{L}$  is a Lebesgue L-system and  $B$  is a correct Borel instance over  $\mathbf{L}$ , then there is a  $\mu_{\mathbf{L}}$ -solution  $f$  for  $B$  such that  $\mathbf{L}[f]$  factors to a Lebesgue L-system.

To this end, suppose that  $\mathbf{L}$  is induced by the  $[0; 1]^2$ -shift action of a countable Borel groupoid  $(R, \Gamma)$  with measure  $\mu \times (\lambda^2)^\Gamma$ , where  $\mu \in \mathcal{P}(R)$ . Consider the L-system  $\mathbf{L}'$  induced by the  $[0; 1]$ -shift action of  $(R, \Gamma)$  with measure  $\mu \times \lambda^\Gamma$ . The map

$$\pi: R \times [0; 1]^\Gamma \times [0; 1]^\Gamma \rightarrow R \times [0; 1]^\Gamma: (r, \vartheta, \omega) \mapsto (r, \vartheta)$$

is  $(R, \Gamma)$ -equivariant and measure-preserving, so, by Lemma 5.14, it is a factor map from  $\mathbf{L}$  to  $\mathbf{L}'$ . Hence, due to Lemma 5.7, there is a correct instance  $B'$  for  $\mathbf{L}'$  such that if  $f'$  is a  $(\mu \times \lambda^\Gamma)$ -solution for  $B'$ , then  $f' \circ \pi$  is a  $(\mu \times (\lambda^2)^\Gamma)$ -solution for  $B$  (modulo an invariant null set). Lemma 5.17 does indeed provide a  $(\mu \times \lambda^\Gamma)$ -solution  $f'$  for  $B'$ , so let  $f := f' \circ \pi$ . Note that, by definition,  $f$  does not depend on the third coordinate. Therefore, by Lemma 5.15,  $\mathbf{L}[f]$  factors to a Lebesgue L-system, as desired.  $\blacksquare$

## 6 The converse of Theorem 5.4 for actions of amenable groups

Theorem 5.4 asserts that if a probability measure-preserving action  $\Gamma \curvearrowright (X, \mu)$  of a countable group  $\Gamma$  factors to the shift action  $\Gamma \curvearrowright ([0; 1]^\Gamma, \lambda^\Gamma)$ , then the induced L-system satisfies the LLL. In this section we show that if  $\Gamma$  is amenable, then the converse implication also holds. In fact, we will show that the existence of a factor map to the shift action in this case is implied by (seemingly) much weaker assumptions than satisfying the LLL.

To state these weaker assumptions, we need some definitions. An instance  $B$  over a set  $X$  is said to be  $\varepsilon$ -correct, where  $\varepsilon \in (0; 1]$ , if it is locally countable and there is a function  $p: \text{dom}(B) \rightarrow [0; 1]$  such that for all  $S \in \text{dom}(B)$ ,

$$\varepsilon^{-|S|} \lambda^S(B_S) \leq \frac{p(S)}{1 - p(S)} \prod_{\substack{S' \in \text{dom}(B): \\ S' \cap S \neq \emptyset}} (1 - p(S')).$$

Hence,  $B$  is 1-correct if and only if it is correct, and

$$B \text{ is } \varepsilon\text{-correct} \implies B \text{ is } \varepsilon'\text{-correct},$$

whenever  $0 < \varepsilon \leq \varepsilon' \leq 1$ .

An instance  $B$  over a set  $X$  is said to be *discrete* if there exist a finite set  $S$ ,  $B' \subseteq [X]_S^{<\infty}$ , and a Borel function  $\varphi: [0; 1] \rightarrow S$  such that  $B$  is the “pullback” of  $B'$  under  $\varphi$ , i.e., for all  $f \in [X]_{[0; 1]}^{<\infty}$ , we have

$$f \in B \iff \varphi \circ f \in B'.$$

Most instances of the LLL that appear in combinatorial applications are discrete; they correspond to randomly choosing a “color” for each  $x \in X$  from  $S$  using the probability measure  $\varphi_*(\lambda)$ . If  $\varphi_*(\lambda)$  is the uniform probability measure on  $S$ , then  $B$  is said to be *uniformly discrete*.

Recall that for a graph  $G$  on  $X$ ,  $\mathcal{I}_G^*$  is the isomorphism structure on  $E_G$  such that a bijection  $\varphi: S_1 \rightarrow S_2$  belongs to  $\mathcal{I}_G^*$  if and only if  $S_1, S_2 \in [E_G]^{<\infty}$  and  $\varphi$  is an isomorphism between  $G|_{S_1}$  and  $G|_{S_2}$ . It is clear from the above definition that an instance  $B$  over  $E_G$  is  $\mathcal{I}_G^*$ -invariant if and only if  $B_S$  is determined by the isomorphism class of the induced subgraph  $G|_S$ . Note that since  $\mathcal{I}_G^* \supseteq \mathcal{I}_G^{<\infty}$ , the class of  $\mathcal{I}_G^*$ -invariant instances is more restrictive than the class of  $\mathcal{I}_G$ -invariant ones.

An instance  $B$  over  $G$ , where  $G$  is a graph on  $X$ , is said to be  *$G$ -connected* if for all  $S \in \text{dom}(B)$ , the graph  $G|_S$  is connected. Most combinatorial applications of the LLL to graph-theoretic problems only consider connected instances.

To put everything together, an instance  $B$  over  $E_G$ , where  $G$  is a graph on  $X$ , is said to be  $(\varepsilon, G)$ -*nice* if it is  $\varepsilon$ -correct, uniformly discrete,  $\mathcal{I}_G^*$ -invariant, and  $G$ -connected.

Recall that for an instance  $B$  over  $X$  and a map  $f: X \rightarrow [0; 1]$ , the  $B$ -defect of  $f$  is the set

$$D(f, B) = \{x \in X : \exists S \in \text{dom}_x(B)(f|_S \in B_S)\}.$$

Now we are ready to state the first version of the converse theorem.

**Theorem 6.1.** *Let  $a: \Gamma \curvearrowright X$  be a free Borel action of a countably infinite amenable group  $\Gamma$  on a standard Borel space  $X$  and let  $\mu \in \mathcal{P}(X)$  be an ergodic invariant probability measure. Let  $S \subseteq \Gamma$  be a finite generating set and let  $G := G(a, S)$ . Then the following conditions are equivalent:*

1. *there exists  $\varepsilon \in (0; 1]$  such that for every  $(\varepsilon, G)$ -nice Borel instance  $B$  over  $E_G$ , there is a Borel map  $f: X \rightarrow [0; 1]$  with  $\mu(D(f, B)) < 1$ ;*
2. *there exists a factor map  $\pi: (X, \mu) \rightarrow ([0; 1]^\Gamma, \lambda^\Gamma)$  to the shift action of  $\Gamma$  on  $[0; 1]^\Gamma$ .*

Note that Theorem 6.1 does not hold for infinite  $S$ . To see this, consider any free Borel action  $a: \Gamma \curvearrowright X$  of a countably infinite amenable group  $\Gamma$  on a standard Borel space  $X$  and let  $\mu \in \mathcal{P}(X)$  be an ergodic invariant probability measure. Take  $S := \Gamma$ . We claim that the L-system  $\mathbf{L} := (X, E_a, \mathcal{I}_{G(a, \Gamma)}, \mu)$  satisfies the LLL, with no restriction on  $a$ . Indeed,  $G(a, \Gamma) = E_a \setminus \Delta_X^2$ , where  $\Delta_X^2 := \{(x, x) : x \in X\}$ . Since, by a theorem of Dye and Ornstein–Weiss (see [22, Theorem 10.7]), all free probability measure-preserving ergodic actions of countable amenable groups are orbit-equivalent,  $\mathbf{L}$  factors to  $\mathbf{L}(b, \lambda^\mathbb{Z})$ , where  $b: \mathbb{Z} \curvearrowright [0; 1]^\mathbb{Z}$  is the shift action of  $\mathbb{Z}$  on  $[0; 1]^\mathbb{Z}$ . By Theorem 5.4,  $\mathbf{L}(b, \lambda^\mathbb{Z})$ , and hence  $\mathbf{L}$ , satisfies the LLL.

However, by remembering slightly more about the action than just the graph  $G(a, S)$ , one can still prove an analog of Theorem 6.1 for infinite  $S$ . An  $(S)$ -*labeled graph* on  $X$  is a family  $G = (G_\gamma)_{\gamma \in S}$  of graphs on  $X$  indexed by the elements of  $S$ , where  $S$  is a countable set. Note

that the sets  $G_\gamma$  are not assumed to be disjoint, i.e., the same edge can receive more than one label. A labeled graph  $G$  on a standard Borel space is said to be *Borel* if each  $G_\gamma$  is Borel. An isomorphism between labeled graphs  $G_1$  and  $G_2$  is required to preserve the labeling, i.e., it has to be an isomorphism between each  $(G_1)_\gamma$  and  $(G_2)_\gamma$ . For an  $S$ -labeled graph  $G$  on  $X$  and a subset  $X' \subseteq X$ , let  $G|X'$  be the  $S$ -labeled graph on  $X'$  given by  $(G|X')_\gamma := G_\gamma|X'$ . For an  $S$ -labeled graph  $G$  on  $X$ , let  $E_G := E_{\bigcup_{\gamma \in S} G_\gamma}$  (note that  $\bigcup_{\gamma \in S} G_\gamma$  is an ordinary graph on  $X$ ). A labeled graph  $G$  is said to be *connected* if  $E_G$  has a single class. The definition of  $\mathcal{I}_G^*$  extends verbatim to the case when  $G$  is labeled, as do the definitions of  $G$ -connected and  $(\varepsilon, G)$ -nice instances over  $E_G$ .

If  $a: \Gamma \curvearrowright X$  is an action of a countable group  $\Gamma$  on a set  $X$  and  $S \subseteq \Gamma$  is a generating set, then  $G_\ell(a, S)$  is the  $S$ -labeled graph on  $X$  given by

$$x(G_\ell(a, S))_\gamma y :\iff x \neq y \text{ and } (\gamma \cdot x = y \text{ or } \gamma \cdot y = x).$$

Note that

$$G(a, S) = \bigcup_{\gamma \in S} (G_\ell(a, S))_\gamma,$$

so we have  $\mathcal{I}_a \subseteq \mathcal{I}_{G_\ell(a, S)}^* \subseteq \mathcal{I}_{G(a, S)}^*$ .

Now we have the following:

**Theorem 6.1'.** *Let  $a: \Gamma \curvearrowright X$  be a free Borel action of a countably infinite amenable group  $\Gamma$  on a standard Borel space  $X$  and let  $\mu \in \mathcal{P}(X)$  be an ergodic invariant probability measure. Let  $S \subseteq \Gamma$  be a generating set and let  $G := G_\ell(a, S)$ . Then the following conditions are equivalent:*

1. *there exists  $\varepsilon \in (0; 1]$  such that for every  $(\varepsilon, G)$ -nice Borel instance  $B$  over  $E_G$ , there is a Borel map  $f: X \rightarrow [0; 1]$  with  $\mu(D(f, B)) < 1$ ;*
2. *there exists a factor map  $\pi: (X, \mu) \rightarrow ([0; 1]^\Gamma, \lambda^\Gamma)$  to the shift action of  $\Gamma$  on  $[0; 1]^\Gamma$ .*

Note that Theorems 6.1 and 6.1' also show that the local finiteness requirement in the statement of Theorem 4.1 is necessary.

## 6.1 Proofs of Theorems 6.1 and 6.1'

### 6.1.1 Outline of the proof

The proofs of Theorems 6.1 and 6.1' are almost identical, so we will present them at the same time. We only need to show the forward direction in both statements, namely that being able to find (approximate) solutions for nice instances of the LLL implies the existence of a factor map to the shift action (the backward direction is handled by Theorem 5.4). In this paragraph we briefly sketch our plan of attack.

For simplicity, assume that  $\Gamma = \mathbb{Z}$  and let  $a: \mathbb{Z} \curvearrowright (X, \mu)$  be a free ergodic probability measure-preserving Borel action of  $\mathbb{Z}$  on a standard probability space  $(X, \mu)$ . There is a simple criterion, called *Sinai's factor theorem*, which determines if there is a factor map  $\pi: (X, \mu) \rightarrow ([0; 1]^\mathbb{Z}, \lambda^\mathbb{Z})$ : Such a map exists if and only if  $a$  has infinite *Kolmogorov–Sinai entropy*. The Kolmogorov–Sinai entropy of  $a$  is defined as follows. First, consider any Borel

function  $f: X \rightarrow S$  to a finite set  $S$ . The *Shannon entropy* of  $f$  measures how “uncertain” the value  $f(x)$  is for  $x$  chosen randomly with respect to  $\mu$ ; formally,

$$h_\mu(f) := - \sum_{c \in S} \mu(f^{-1}(c)) \log_2 \mu(f^{-1}(c)).$$

Now the action comes into play: Given  $x \in X$  and  $n \in \mathbb{N}$ , we record the values  $f((-n) \cdot x)$ ,  $f((-n+1) \cdot x)$ ,  $\dots$ ,  $f(n \cdot x)$ ; let  $f_n: X \rightarrow S^{2n+1}$  denote the corresponding function. We can compute the *average* amount of uncertainty in  $f_n(x)$  per symbol; in other words, we can look at the quantity  $h_\mu(f_n(x))/(2n+1)$ . It turns out that, as  $n$  grows,  $h_\mu(f_n(x))/(2n+1)$  decreases, so there is a limit

$$H_\mu(a, f) := \lim_{n \rightarrow \infty} \frac{h_\mu(f_n)}{2n+1}.$$

This limit is called the *Kolmogorov–Sinai entropy* of  $f$  with respect to  $a$ . The *Kolmogorov–Sinai entropy* of the action  $a$  itself measures the “maximum level of uncertainty” that can be achieved with respect to  $a$ ; formally, it is defined to be  $H_\mu(a) := \sup_f H_\mu(a, f)$ . As we mentioned before,  $a$  factors to the  $[0; 1]$ -shift action if and only if  $H_\mu(a) = \infty$ .

How can we use the LLL to prove that  $H_\mu(a) = \infty$ ? By definition, we have to exhibit functions  $f$  with arbitrarily large values of  $H_\mu(a, f)$ . But  $H_\mu(a, f)$  is a “global” parameter—it is defined in terms of the measures of some subsets of  $X$ —while instances of the LLL can only put “local” constraints on  $f$ . Notice, however, that high value of  $H_\mu(a, f)$  indicates that the functions  $f_n$  behave very “randomly” or “unpredictably.” Thus, what we need is a way to measure “randomness” or “unpredictability” deterministically, which we can apply to the values of  $f_n$  at each point instead of looking at the function  $f_n$  as a whole.

There is indeed a convenient deterministic analog of Shannon’s entropy, namely the so-called *Kolmogorov complexity*. Roughly speaking, a finite sequence  $w$  of symbols has high Kolmogorov complexity if there is no way to encode it by a significantly shorter sequence. Our instance of the LLL will require  $f_n(x)$  to have high Kolmogorov complexity for all  $n \in \mathbb{N}$  and  $x \in X$ . We will show that solving this instance, even partially, guarantees that  $H_\mu(a, f)$  is also high.

The structure of the rest of this section is as follows. In §6.1.2 we list all the necessary definitions and preliminary results regarding the structure of amenable groups, Kolmogorov–Sinai entropy of their actions (including the version of Sinai’s factor theorem with a general amenable group in place of  $\mathbb{Z}$ ), and Kolmogorov complexity. In §6.1.3 we prove the main lemma that connects Kolmogorov complexity and Kolmogorov–Sinai entropy. Finally, §6.1.4 completes the proof by constructing a series of  $(\varepsilon, G)$ -nice instances whose solutions necessarily have high Kolmogorov complexity and hence high Kolmogorov–Sinai entropy.

### 6.1.2 Preliminaries

**Background on amenable groups** For a group  $\Gamma$ , subsets  $S, T \subseteq \Gamma$ , and an element  $\gamma \in \Gamma$ , let

$$\begin{aligned} \gamma S &:= \{\gamma\delta : \delta \in S\}; \\ S\gamma &:= \{\delta\gamma : \delta \in S\}; \\ \text{and } ST &:= \{\delta_1\delta_2 : \delta_1 \in S, \delta_2 \in T\}. \end{aligned}$$

Recall that a countable group  $\Gamma$  is called *amenable* if it admits a *Følner sequence*, i.e., a sequence  $(\Phi_n)_{n=0}^\infty$  of nonempty finite subsets of  $\Gamma$  such that for all  $\gamma \in \Gamma$ ,

$$\lim_{n \rightarrow \infty} \frac{|\gamma \Phi_n \Delta \Phi_n|}{|\Phi_n|} = 0, \quad (6.1)$$

where  $\Delta$  denotes symmetric difference of sets. Note that if  $S \subseteq \Gamma$  is a generating set and (6.1) holds for all  $\gamma \in S$ , then  $(\Phi_n)_{n=0}^\infty$  is a Følner sequence (see [22, Remark 5.12]).

**Proposition 6.2.** *Suppose  $\Gamma$  is a countably infinite amenable group and  $S \subseteq \Gamma$  is a generating set. Let  $G := \text{Cay}(\Gamma, S)$  be the corresponding Cayley graph. Then  $\Gamma$  admits a Følner sequence  $(\Phi_n)_{n=0}^\infty$  such that each finite graph  $G|_{\Phi_n}$  is connected.*

*Proof.* Let  $S = \{\gamma_0, \gamma_1, \dots\}$  and let  $(\Phi_n)_{n=0}^\infty$  be a Følner sequence for  $\Gamma$ . By passing to a subsequence if necessary, we can arrange that for all  $n \in \mathbb{N}$ ,

$$\sum_{i=0}^{\min\{|S|, n\}} \frac{|\gamma_i \Phi_n \Delta \Phi_n|}{|\Phi_n|} \leq \frac{1}{n}. \quad (6.2)$$

Suppose  $G|_{\Phi_n}$  has  $k_n$  connected components and let  $\Phi_{n,1}, \dots, \Phi_{n,k_n} \subseteq \Phi_n$  be their vertex sets. Note that, since  $\gamma_i \Phi_{n,j_1} \cap \Phi_{n,j_2} = \emptyset$  for all  $j_1 \neq j_2$ , we have

$$\gamma_i \Phi_n \Delta \Phi_n = \bigcup_{j=1}^{k_n} (\gamma_i \Phi_{n,j} \Delta \Phi_{n,j}), \quad (6.3)$$

and the union on the right-hand side of (6.3) is disjoint. Therefore,

$$|\gamma_i \Phi_n \Delta \Phi_n| = \sum_{j=1}^{k_n} |\gamma_i \Phi_{n,j} \Delta \Phi_{n,j}|$$

Thus,

$$\sum_{i=0}^{\min\{|S|, n\}} \frac{|\gamma_i \Phi_n \Delta \Phi_n|}{|\Phi_n|} = \frac{\sum_{j=1}^{k_n} \sum_{i=0}^{\min\{|S|, n\}} |\gamma_i \Phi_{n,j} \Delta \Phi_{n,j}|}{\sum_{j=1}^{k_n} |\Phi_{n,j}|}.$$

If for all  $1 \leq j \leq k_n$ , we have

$$\sum_{i=0}^{\min\{|S|, n\}} \frac{|\gamma_i \Phi_{n,j} \Delta \Phi_{n,j}|}{|\Phi_{n,j}|} > \frac{1}{n},$$

then

$$\frac{\sum_{j=1}^{k_n} \sum_{i=0}^{\min\{|S|, n\}} |\gamma_i \Phi_{n,j} \Delta \Phi_{n,j}|}{\sum_{j=1}^{k_n} |\Phi_{n,j}|} > \frac{\sum_{j=1}^{k_n} \frac{1}{n} |\Phi_{n,j}|}{\sum_{j=1}^{k_n} |\Phi_{n,j}|} = \frac{1}{n},$$

which contradicts (6.2). Hence, for some  $1 \leq j_n \leq k_n$ ,

$$\sum_{i=0}^{\min\{|S|, n\}} \frac{|\gamma_i \Phi_{n,j_n} \Delta \Phi_{n,j_n}|}{|\Phi_{n,j_n}|} \leq \frac{1}{n}.$$

Now  $(\Phi_{n,j_n})_{n=0}^\infty$  is a Følner sequence consisting of connected sets. ■

**Corollary 6.3.** *Suppose  $\Gamma$  is a countably infinite amenable group and  $S \subseteq \Gamma$  is a generating set. Let  $G := \text{Cay}(\Gamma, S)$  be the corresponding Cayley graph. Then  $\Gamma$  admits a Følner sequence  $(\Phi_n)_{n=0}^\infty$  such that:*

1. *for each  $n \in \mathbb{N}$ , the graph  $\text{Cay}(\Gamma, S)|_{\Phi_n}$  is connected;*
2.  *$\mathbf{1} \in \Phi_0 \subset \Phi_1 \subset \dots$ , where  $\mathbf{1}$  is the identity element of  $\Gamma$ ;*
3.  *$\bigcup_{n=0}^\infty \Phi_n = \Gamma$ ;*
4.  *$\lim_{n \rightarrow \infty} |\Phi_n| / \log_2 n = \infty$ .*

*Proof.* Proposition 6.2 provides a Følner sequence  $(\Phi_n)_{n=0}^\infty$  satisfying the first condition. Since  $\Gamma$  is infinite,  $|\Phi_n| \rightarrow \infty$  as  $n \rightarrow \infty$ . If  $\mathbf{1} \notin \Phi_n$  for some  $n \in \mathbb{N}$ , then choose any  $\gamma \in \Phi_n$  and replace  $\Phi_n$  with  $\Phi_n \gamma^{-1}$ . We construct a new sequence  $(\Phi'_n)_{n=0}^\infty$  recursively. Let  $S =: \{\gamma_0, \gamma_1, \dots\}$  and for  $n \in \mathbb{N}$ , let

$$S_n := \begin{cases} S & \text{if } |S| \leq n; \\ \{\gamma_0, \dots, \gamma_n\} & \text{otherwise.} \end{cases}$$

Let  $B_n$  be the set of all elements of  $\Gamma$  that can be expressed as products of at most  $n$  elements of  $S_n$ . Let  $\Phi'_0 := \Phi_0$ . On step  $n+1$ , choose  $N$  large enough so that

$$|\Phi_N| > n \left( \sum_{i=0}^n |\Phi'_i| + |B_n| + \log_2 n \right),$$

and let

$$\Phi'_{n+1} := \Phi_N \cup \bigcup_{i=0}^n \Phi'_i \cup B_n.$$

Clearly,  $(\Phi'_n)_{n=0}^\infty$  is a Følner sequence satisfying all the requirements. ■

We will use a result of Ornstein–Weiss on the existence of quasi-tilings in amenable groups. A family  $(A_i)_{i=1}^k$  of finite sets is said to be  $\varepsilon$ -disjoint, where  $\varepsilon > 0$ , if there exist disjoint subsets  $B_1 \subseteq A_1, \dots, B_k \subseteq A_k$  such that for all  $1 \leq i \leq k$ ,

$$|B_i| \geq (1 - \varepsilon)|A_i|.$$

A finite set  $A$  is  $(1 - \varepsilon)$ -covered by  $(A_i)_{i=1}^k$  if

$$\left| \bigcup_{i=1}^k A_i \cap A \right| \geq (1 - \varepsilon)|A|.$$

Let  $\Gamma$  be a countable group and let  $A, A_1, \dots, A_k$  be finite subsets of  $\Gamma$ . An  $\varepsilon$ -quasi-tiling of  $A$  by  $(A_i)_{i=1}^k$  is a family  $(C_i)_{i=1}^k$  of finite subsets of  $\Gamma$  such that:

1. for each  $1 \leq i \leq k$ ,  $A_i C_i \subseteq A$  and the family of sets  $(A_i \gamma)_{\gamma \in C_i}$  is  $\varepsilon$ -disjoint;
2. the sets  $(A_i C_i)_{i=1}^k$  are disjoint;

3.  $A$  is  $(1 - \varepsilon)$ -covered by the family  $(A_i C_i)_{i=1}^k$ .

**Theorem 6.4** (Ornstein–Weiss [34]; see also [41, Theorem 2.6] and [42, Proposition 2.3]). *Let  $\Gamma$  be a countable amenable group and let  $(\Phi_n)_{n=0}^\infty$  be a Følner sequence in  $\Gamma$ . Then for all  $\varepsilon > 0$  and  $n \in \mathbb{N}$ , there exist  $\ell_1, \dots, \ell_k, m_0 \in \mathbb{N}$  with  $n \leq \ell_1 < \ell_2 < \dots < \ell_k$  such that for each  $m \geq m_0$ , there exists an  $\varepsilon$ -quasi-tiling of  $\Phi_m$  by  $(\Phi_{\ell_i})_{i=1}^k$ .*

**Background on Kolmogorov–Sinai entropy** An important invariant of an amenable probability measure-preserving system is its Kolmogorov–Sinai entropy. It is usually defined in terms of finite Borel partitions; however, for our purposes it will be more convenient to define it in terms of Borel functions with finite range (the two notions are, of course, equivalent).

Let  $X$  be a standard Borel space and let  $\mu \in \mathcal{P}(X)$ . A *finite coloring* of  $X$  is a function  $f: X \rightarrow S$ , where  $S$  is a finite set. The *Shannon entropy* of a Borel finite coloring  $f: X \rightarrow S$  (notation:  $h_\mu(f)$ ) is defined to be

$$h_\mu(f) := - \sum_{c \in S} \mu(f^{-1}(c)) \log_2 \mu(f^{-1}(c)).$$

Here we adopt the convention that  $0 \cdot \log_2 0 = 0$ . Note that  $0 \leq h_\mu(f) \leq \log_2 |S|$ .

Suppose that  $a: \Gamma \curvearrowright (X, \mu)$  is a probability measure-preserving Borel action of a countable amenable group  $\Gamma$ . For a finite coloring  $f: X \rightarrow S$  and a set  $\Phi \in [\Gamma]^{<\infty}$ , let  $f^\Phi: X \rightarrow S^\Phi$  be the finite coloring such that for all  $x \in X$  and  $\gamma \in \Phi$ ,

$$f^\Phi(x)(\gamma) := f(\gamma \cdot x).$$

The *Kolmogorov–Sinai entropy* of a Borel finite coloring  $f$  with respect to  $a$  is defined by

$$H_\mu(a, f) := \lim_{n \rightarrow \infty} \frac{h_\mu(f^{\Phi_n})}{|\Phi_n|}, \quad (6.4)$$

where  $(\Phi_n)_{n=0}^\infty$  is a Følner sequence in  $\Gamma$ . Due to a fundamental result of Ornstein and Weiss [34], the limit in (6.4) always exists and is independent of  $(\Phi_n)_{n=0}^\infty$ . Note that, again,  $0 \leq H_\mu(a, f) \leq \log_2 |S|$ , where  $S$  is the range of  $f$ . The *Kolmogorov–Sinai entropy* of  $a$  is defined as follows:

$$H_\mu(a) := \sup\{H_\mu(a, f) : f \text{ is a Borel finite coloring of } X\}.$$

We will use the following special case of a generalization of Sinai’s factor theorem to actions of arbitrary amenable groups due to Ornstein and Weiss.

**Theorem 6.5** (Ornstein–Weiss [34]). *Let  $a: \Gamma \curvearrowright X$  be a free Borel action of a countably infinite amenable group  $\Gamma$  on a standard Borel space  $X$  and let  $\mu \in \mathcal{P}(X)$  be an ergodic invariant probability measure. Suppose that  $H_\mu(a) = \infty$ . Then there exists a factor map  $\pi: (X, \mu) \rightarrow ([0; 1]^\Gamma, \lambda^\Gamma)$  to the shift action of  $\Gamma$  on  $[0; 1]^\Gamma$ .*



**Background on Kolmogorov complexity** We will use some basic properties of Kolmogorov complexity. For sets  $X, Y$ , a *partial function*  $f$  from  $X$  to  $Y$  (notation:  $f: X \dashrightarrow Y$ ) is a function of the form  $f: X' \rightarrow Y$ , where  $X' \subseteq X$ . Let  $\{0, 1\}^*$  denote the set of all finite sequences of zeroes and ones (including the empty one). For  $w \in \{0, 1\}^*$ , let  $|w|$  denote the length of  $w$ . For a partial function  $D: \{0, 1\}^* \dashrightarrow \{0, 1\}^*$ , define the map  $K_D: \{0, 1\}^* \rightarrow \mathbb{N} \cup \{\infty\}$  by

$$K_D(x) := \inf\{|w| : D(w) = x\}.$$

For two partial functions  $D_1, D_2: \{0, 1\}^* \dashrightarrow \{0, 1\}^*$ , we say that  $D_1$  *minorizes*  $D_2$  (notation:  $D_1 \leq_K D_2$ ) if there is a constant  $c \in \mathbb{N}$  such that for all  $x \in \{0, 1\}^*$ ,

$$K_{D_1}(x) \leq K_{D_2}(x) + c.$$

Clearly,  $\leq_K$  is a preorder on the set of all partial maps from  $\{0, 1\}^*$  to itself. If  $\mathcal{C}$  is a class of partial functions from  $\{0, 1\}^*$  to itself, then  $D \in \mathcal{C}$  is *optimal* in  $\mathcal{C}$  if for all  $D' \in \mathcal{C}$ ,  $D \leq_K D'$ .

We restrict our attention to the class of partial maps  $D: \{0, 1\}^* \dashrightarrow \{0, 1\}^*$  that are computable relative to some oracle  $\mathbb{O}$  (denote this class by  $\mathcal{C}_{\mathbb{O}}$ ). A cornerstone of the theory of Kolmogorov complexity is the following observation:

**Theorem 6.6** (Solomonoff–Kolmogorov; see [28, Lemma 2.1.1], [39, Theorem 1]). *Fix an oracle  $\mathbb{O}$ . There exists a map  $D \in \mathcal{C}_{\mathbb{O}}$  that is optimal in  $\mathcal{C}_{\mathbb{O}}$ .*

Using Theorem 6.6, we can define the *Kolmogorov complexity* of a word  $x \in \{0, 1\}^*$  relative to an oracle  $\mathbb{O}$  to be

$$K_{\mathbb{O}}(x) := K_D(x)$$

for some fixed optimal  $D \in \mathcal{C}_{\mathbb{O}}$ . Note that if  $D, D' \in \mathcal{C}_{\mathbb{O}}$  are two optimal functions, then there is a constant  $c \in \mathbb{N}$  such that  $|K_D(x) - K_{D'}(x)| \leq c$  for all  $x \in \{0, 1\}^*$ ; in this sense, the value  $K_{\mathbb{O}}(x)$  is defined up to an additive constant.

The following property of Kolmogorov complexity will play a key role in our argument.

**Proposition 6.7.** *Fix an oracle  $\mathbb{O}$ . Let  $c, n \in \mathbb{N}$  and let  $\nu_n$  denote the uniform probability measure on  $\{0, 1\}^n$ . Then*

$$\nu_n(\{x \in \{0, 1\}^n : K_{\mathbb{O}}(x) \leq n - c\}) < 2^{-c+1}.$$

*Proof.* Let  $D \in \mathcal{C}_{\mathbb{O}}$  be the optimal function used in the definition of Kolmogorov complexity relative to  $\mathbb{O}$ . There are exactly  $2^{n-c+1} - 1$  sequences of zeroes and ones of length at most  $n - c$ , so there can be at most  $2^{n-c+1} - 1$  words  $x \in \{0, 1\}^n$  with  $K_{\mathbb{O}}(x) = K_D(x) \leq n - c$ . Since there are  $2^n$  words of length  $n$ , we get

$$\nu_n(\{x \in \{0, 1\}^n : K_{\mathbb{O}}(x) \leq n - c\}) \leq \frac{2^{n-c+1} - 1}{2^n} = 2^{-c+1} - 2^{-n} < 2^{-c+1},$$

as desired. ■

### 6.1.3 Kolmogorov complexity vs. Kolmogorov–Sinai entropy

For the rest of this section, we fix a countably infinite amenable group  $\Gamma$ , a generating set  $S \subseteq \Gamma$ , a standard Borel space  $X$ , a free Borel action  $a: \Gamma \curvearrowright X$ , and an ergodic invariant probability measure  $\mu \in \mathcal{P}(X)$ . We also fix a Følner sequence  $(\Phi_n)_{n=0}^\infty$  in  $\Gamma$  satisfying the conditions of Corollary 6.3, i.e., such that:

1. for each  $n \in \mathbb{N}$ , the graph  $\text{Cay}(\Gamma, S)|_{\Phi_n}$  is connected;
2.  $\mathbf{1} \in \Phi_0 \subset \Phi_1 \subset \dots$ , where  $\mathbf{1}$  is the identity element of  $\Gamma$ ;
3.  $\bigcup_{n=0}^\infty \Phi_n = \Gamma$ ;
4.  $\lim_{n \rightarrow \infty} |\Phi_n| / \log_2 n = \infty$ .

Let  $\mathbb{O}$  be an oracle relative to which the following data are computable:

1. the group structure of  $\Gamma$  and a fixed linear ordering  $<$  on  $\Gamma$  (we can assume, for instance, that the ground set of  $\Gamma$  is  $\mathbb{N}$ );
2. the sequence  $(\Phi_n)_{n=0}^\infty$  (meaning that the set of pairs  $\{(\gamma, n) \in \Gamma \times \mathbb{N} : \gamma \in \Phi_n\}$  and the sequence  $(|\Phi_n|)_{n=0}^\infty$  are decidable relative to  $\mathbb{O}$ ).

Let  $s \in \mathbb{N}$ . For a set  $\Phi \in [\Gamma]^{<\infty}$ , the ordering on  $\Gamma$  induces a bijection between  $(\{0, 1\}^s)^\Phi$  and  $\{0, 1\}^{s|\Phi|}$ ; namely, if  $\Phi = \{\gamma_1, \dots, \gamma_{|\Phi|}\}$  with  $\gamma_1 < \dots < \gamma_{|\Phi|}$  and  $w \in (\{0, 1\}^s)^\Phi$ , then the corresponding element of  $\{0, 1\}^{s|\Phi|}$  is

$$w(\gamma_1)^\wedge \dots^\wedge w(\gamma_{|\Phi|}),$$

where  $^\wedge$  denotes concatenation of finite sequences. Using this identification, we can talk about Kolmogorov complexity of elements of  $(\{0, 1\}^s)^\Phi$ . Given  $f: X \rightarrow \{0, 1\}^s$ ,  $n \in \mathbb{N}$ , and  $x \in X$ , let

$$f_n(x) := f^{\Phi_n}(x).$$

Note that for a Borel finite coloring  $f$ , the map  $\mathbb{N} \times X \rightarrow \mathbb{N}: (n, x) \mapsto K_{\mathbb{O}}(f_n(x))$  is Borel.

The following lemma connects Kolmogorov complexity and Kolmogorov–Sinai entropy:

**Lemma 6.8** (High complexity  $\implies$  high entropy). *Let  $s \in \mathbb{N}$  and let  $f: X \rightarrow \{0, 1\}^s$  be a Borel finite coloring of  $X$ . Then*

$$\limsup_{m \rightarrow \infty} \int_X \frac{K_{\mathbb{O}}(f_m(x))}{|\Phi_m|} d\mu(x) \leq H_\mu(a, f).$$

*Proof.* The proof is inspired by the work of Brudno [7], establishing a relationship between Kolmogorov complexity and Kolmogorov–Sinai entropy in the case of  $\mathbb{Z}$ -actions (see also [32] for an extension of Brudno’s theory to a wider class of amenable groups).

Fix  $\varepsilon \in (0, 1)$ . Choose  $n \in \mathbb{N}$  large enough so that for all  $\ell \geq n$ ,

$$\frac{h_\mu(f_\ell)}{|\Phi_\ell|} \leq H_\mu(a, f) + \varepsilon.$$

For most of the proof  $n$  and  $s$  will be treated as fixed constants. In particular, implied constants in the Bachmann–Landau asymptotic notation can depend on  $n$  and  $s$ . By Theorem 6.4, we can choose  $k, \ell_1, \dots, \ell_k, m_0 \in \mathbb{N}$  such that  $n \leq \ell_1 < \dots < \ell_k$  and for every  $m \geq m_0$ , there exists an  $\varepsilon$ -quasi-tiling of  $\Phi_m$  by  $(\Phi_{\ell_i})_{i=1}^k$ . For  $m \geq m_0$ , let  $(C_{m,i})_{i=1}^k$  be an  $\varepsilon$ -quasi-tiling of  $\Phi_m$  by  $(\Phi_{\ell_i})_{i=1}^k$  chosen in such a way that the map  $m \mapsto (C_{m,i})_{i=1}^k$  is computable relative to  $\mathbb{O}$ .

We will now devise a binary code for pairs of the form  $(m, w)$ , where  $m \geq m_0$  and  $w \in (\{0, 1\}^s)^{\Phi_m}$ . The decoding procedure for this code will be computable relative to  $\mathbb{O}$ , so its length will provide an upper bound on the Kolmogorov complexity of  $w$  (modulo an additive constant).

Let  $c_0(m)$  be the sequence of  $\lceil \log_2 m \rceil$  ones followed by a single zero. Let  $c_1(m)$  be any fixed binary code for  $m$  of length exactly  $\lceil \log_2 m \rceil$ . Note that for any  $c \in \{0, 1\}^{<\infty}$ , the pair  $(m, c)$  is uniquely determined by  $c_0(m) \frown c_1(m) \frown c$ . Also note that  $|c_0(m) \frown c_1(m)| \leq 2 \log_2 m + O(1) = o_{m \rightarrow \infty}(|\Phi_m|)$ .

Consider the set

$$\Lambda_m := \Phi_m \setminus \bigcup_{i=1}^k \Phi_{\ell_i} C_{m,i}.$$

We can view  $w|_{\Lambda_m}$  as a binary word of length  $s|\Lambda_m|$ . Denote this word by  $c_2(m, w)$ . Note that the length of  $c_2(m, w)$  is determined by  $m$  and satisfies

$$|c_2(m, w)| = s|\Lambda_m| \leq \varepsilon s|\Phi_m|,$$

since  $\Phi_m$  is  $(1 - \varepsilon)$ -covered by the family  $(\Phi_{\ell_i} C_{m,i})_{i=1}^k$ .

For each  $1 \leq i \leq k$  and  $\gamma \in C_{m,i}$ , we can view

$$w_{i,\gamma} := w|_{(\Phi_{\ell_i} \gamma)}$$

as a binary word of length  $s|\Phi_{\ell_i}|$ . For each binary word  $u$  of length  $s|\Phi_{\ell_i}|$ , let  $\eta_{i,u}(m, w)$  be the frequency of  $u$  among the words of the form  $w_{i,\gamma}$ , i.e., let

$$\eta_{i,u}(m, w) := |\{\gamma \in C_{m,i} : w_{i,\gamma} = u\}|.$$

Note that, by definition,

$$\sum_u \eta_{i,u}(m, w) = |C_{m,i}|, \tag{6.5}$$

where the summation is over all binary words of length  $s|\Phi_{\ell_i}|$ . Since  $0 \leq \eta_{i,u}(m, w) \leq |C_{m,i}|$ , we can encode  $\eta_{i,u}(m, w)$  by a binary word  $c_3(m, w, i, u)$  of length exactly  $\lceil \log_2(|C_{m,i}| + 1) \rceil \leq \log_2 |C_{m,i}| + O(1) \leq \log_2 |\Phi_m| + O(1)$ , so the length of  $c_3(m, w, i, u)$  is determined by  $m$ . Let  $c_3(m, w, i)$  denote the concatenation of all the words of the form  $c_3(m, w, i, u)$ , where  $u$  is a binary word of length  $s|\Phi_{\ell_i}|$ , and let  $c_3(m, w) := c_3(m, w, 1) \frown \dots \frown c_3(m, w, k)$ . The length of  $c_3(m, w)$  is at most  $O(\log_2 |\Phi_m|) = o_{m \rightarrow \infty}(|\Phi_m|)$ .

Now we consider the word

$$w_i := w|_{(\Phi_{\ell_i} C_{m,i})}.$$

Since  $w_i$  is determined by  $(w_{i,\gamma} : \gamma \in C_{m,i})$ , there are at most

$$\frac{|C_{m,i}|!}{\prod_u \eta_{i,u}(m, w)!}$$

options for  $w_i$ , where the product is over all binary words of length  $s|\Phi_{\ell_i}|$ . Note that, due to Stirling's formula and equation (6.5), we have

$$\log_2 \left( \frac{|C_{m,i}|!}{\prod_u \eta_{i,u}(m, w)!} \right) \leq -|C_{m,i}| \sum_u \frac{\eta_{i,u}(m, w)}{|C_{m,i}|} \log_2 \left( \frac{\eta_{i,u}(m, w)}{|C_{m,i}|} \right).$$

Thus, provided that  $m$  and all the  $\eta_{i,u}(m, w)$ 's are given, we can encode  $w_i$  by a word  $c_4(m, w, i)$  of length

$$\left\lceil \log_2 \left( \frac{|C_{m,i}|!}{\prod_u \eta_{i,u}(m, w)!} \right) \right\rceil \leq -|C_{m,i}| \sum_u \frac{\eta_{i,u}(m, w)}{|C_{m,i}|} \log_2 \left( \frac{\eta_{i,u}(m, w)}{|C_{m,i}|} \right) + O(1).$$

Let  $c_4(m, w) := c_4(m, w, 1) \frown \dots \frown c_4(m, w, k)$ . Then the length of  $c_4(m, w)$  is at most

$$- \sum_{i=1}^k |C_{m,i}| \sum_u \frac{\eta_{i,u}(m, w)}{|C_{m,i}|} \log_2 \left( \frac{\eta_{i,u}(m, w)}{|C_{m,i}|} \right) + o_{m \rightarrow \infty}(|\Phi_m|).$$

Our code for  $(m, w)$  is

$$\text{code}(m, w) := c_0(m) \frown c_1(m) \frown c_2(m, w) \frown c_3(m, w) \frown c_4(m, w).$$

It is clear that  $\text{code}(m, w)$  uniquely determines  $m$  and  $w$  and, moreover, the map

$$\text{code}(m, w) \mapsto (m, w)$$

is computable relative to  $\mathbb{O}$ . Combining the estimates for the lengths of  $c_0(m)$ ,  $c_1(m)$ ,  $c_2(m, w)$ ,  $c_3(m, w)$ , and  $c_4(m, w)$ , we get

$$|\text{code}(m, w)| \leq \varepsilon s |\Phi_m| - \sum_{i=1}^k |C_{m,i}| \sum_u \frac{\eta_{i,u}(m, w)}{|C_{m,i}|} \log_2 \left( \frac{\eta_{i,u}(m, w)}{|C_{m,i}|} \right) + o_{m \rightarrow \infty}(|\Phi_m|).$$

Since  $K_{\mathbb{O}}(w) \leq |\text{code}(m, w)| + O(1)$ , the same asymptotic upper bound holds for  $K_{\mathbb{O}}(w)$  as well.

Applying this analysis to a point  $x \in X$ , we obtain

$$\frac{K_{\mathbb{O}}(f_m(x))}{|\Phi_m|} \leq \varepsilon s - \sum_{i=1}^k \frac{|C_{m,i}|}{|\Phi_m|} \sum_u \frac{\eta_{i,u}(m, f_m(x))}{|C_{m,i}|} \log_2 \left( \frac{\eta_{i,u}(m, f_m(x))}{|C_{m,i}|} \right) + o_{m \rightarrow \infty}(1). \quad (6.6)$$

**Claim 6.8.1.** *For each  $m \geq m_0$ ,  $1 \leq i \leq k$ , and a binary word  $u$  of length  $s|\Phi_{\ell_i}|$ , we have*

$$\int_X \frac{\eta_{i,u}(m, f_m(x))}{|C_{m,i}|} d\mu(x) = \mu(f_{\ell_i}^{-1}(u)).$$

*Proof.* Recall that, by definition,

$$\eta_{i,u}(m, w) = |\{\gamma \in C_{m,i} : w_{i,\gamma} = u\}| = |\{\gamma \in C_{m,i} : w|(\Phi_{\ell_i}\gamma) = u\}|.$$

Notice that

$$f_m(x)|(\Phi_{\ell_i}\gamma) = f_{\ell_i}(\gamma \cdot x),$$

so

$$\begin{aligned}\eta_{i,u}(m, f_m(x)) &= |\{\gamma \in C_{m,i} : f_m(x)|(\Phi_{\ell_i}\gamma) = u\}| \\ &= |\{\gamma \in C_{m,i} : f_{\ell_i}(\gamma \cdot x) = u\}|.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_X \frac{\eta_{i,u}(m, f_m(x))}{|C_{m,i}|} d\mu(x) &= \frac{1}{|C_{m,i}|} \int_X |\{\gamma \in C_{m,i} : f_{\ell_i}(\gamma \cdot x) = u\}| d\mu(x) \\ &= \frac{1}{|C_{m,i}|} \int_X |\{\gamma \in C_{m,i} : f_{\ell_i}(x) = u\}| d\mu(x) \\ &= \frac{1}{|C_{m,i}|} \cdot |C_{m,i}| \cdot \mu(f_{\ell_i}^{-1}(u)) = \mu(f_{\ell_i}^{-1}(u)),\end{aligned}$$

where the second equality holds since  $\mu$  is invariant.  $\dashv$

Note that the function  $\alpha \mapsto -\alpha \log_2 \alpha$  is concave for  $0 \leq \alpha \leq 1$ , so, by Claim 6.8.1, for  $1 \leq i \leq k$ , we have

$$\begin{aligned}& - \int_X \sum_u \frac{\eta_{i,u}(m, f_m(x))}{|C_{m,i}|} \log_2 \left( \frac{\eta_{i,u}(m, f_m(x))}{|C_{m,i}|} \right) d\mu(x) \\ & \leq - \sum_u \int_X \frac{\eta_{i,u}(m, f_m(x))}{|C_{m,i}|} d\mu(x) \log_2 \left( \int_X \frac{\eta_{i,u}(m, f_m(x))}{|C_{m,i}|} d\mu(x) \right) \\ & = - \sum_u \mu(f_{\ell_i}^{-1}(u)) \log_2 \mu(f_{\ell_i}^{-1}(u)) \\ & = h_\mu(f_{\ell_i}).\end{aligned}$$

Combining this with (6.6) gives

$$\int_X \frac{K_{\mathbb{O}}(f_m(x))}{|\Phi_m|} d\mu(x) \leq \varepsilon s + \sum_{i=1}^k \frac{|C_{m,i}|}{|\Phi_m|} h_\mu(f_{\ell_i}) + o_{m \rightarrow \infty}(1).$$

Recall that, by the choice of  $n$ ,

$$\frac{h_\mu(f_\ell)}{|\Phi_\ell|} \leq H_\mu(a, f) + \varepsilon$$

for all  $\ell \geq n$ . Since  $n \leq \ell_1 < \dots < \ell_k$ , we get

$$\int_X \frac{K_{\mathbb{O}}(f_m(x))}{|\Phi_m|} d\mu(x) \leq \varepsilon s + \sum_{i=1}^k \frac{|C_{m,i}|}{|\Phi_m|} |\Phi_{\ell_i}| (H_\mu(a, f) + \varepsilon) + o_{m \rightarrow \infty}(1).$$

Since the family  $(\Phi_{\ell_i} C_{m,i})_{i=1}^k$  is disjoint, and for each  $1 \leq i \leq k$ , the family  $(\Phi_{\ell_i} \gamma)_{\gamma \in C_{m,i}}$  is  $\varepsilon$ -disjoint, we have

$$|\Phi_m| \geq \sum_{i=1}^k |\Phi_{\ell_i} C_{m,i}| \geq (1 - \varepsilon) \sum_{i=1}^k |\Phi_{\ell_i}| |C_{m,i}|,$$

so

$$\int_X \frac{K_{\mathbb{O}}(f_m(x))}{|\Phi_m|} d\mu(x) \leq \varepsilon s + \frac{1}{1 - \varepsilon} (H_{\mu}(a, f) + \varepsilon) + o_{m \rightarrow \infty}(1). \quad (6.7)$$

Since (6.7) holds for every  $\varepsilon \in (0; 1)$  and for every sufficiently large  $m$ , we finally obtain

$$\int_X \frac{K_{\mathbb{O}}(f_m(x))}{|\Phi_m|} d\mu(x) \leq H_{\mu}(a, f) + o_{m \rightarrow \infty}(1),$$

as desired. ■

#### 6.1.4 Building the instances

Let

$$G := \begin{cases} G(a, S) & \text{if } S \text{ is finite;} \\ G_{\ell}(a, S) & \text{if } S \text{ is infinite.} \end{cases}$$

If  $S$  is finite, let  $\text{Cay}(\Gamma, S)$  denote the corresponding unlabeled Cayley graph; otherwise, the graph  $\text{Cay}(\Gamma, S)$  is assumed to be  $S$ -labeled. Let  $\varepsilon \in (0; 1]$  be such that for every  $(\varepsilon, G)$ -nice Borel instance  $B$  over  $E_G$ , there is a Borel map  $f: X \rightarrow [0; 1]$  with  $\mu(D(f, B)) < 1$ . Our goal is to show that  $H_{\mu}(a) = \infty$ .

For  $s, t \in \mathbb{N}$  with  $s \geq t$ , we will construct a Borel instance  $B(s, t)$  over  $E_G$ . It will be uniformly discrete; we identify it with a Borel subset of  $[E_G]_{\{0,1\}^s}^{<\infty}$  instead of the usual  $[E_G]_{[0;1]}^{<\infty}$ .

For  $n \in \mathbb{N}$ , let  $G_n := \text{Cay}(\Gamma, S)|\Phi_n$ . Note that, by the choice of  $(\Phi_n)_{n=0}^{\infty}$ , each  $G_n$  is connected. For  $T \in [E_G]^{<\infty}$ , a function  $f: T \rightarrow \{0, 1\}^s$  belongs to  $B(s, t)$  if and only if, for some  $n \in \mathbb{N}$ , the graphs  $G_n$  and  $G|T$  are isomorphic and there is an isomorphism  $\varphi: \Phi_n \rightarrow T$  such that

$$K_{\mathbb{O}}(f \circ \varphi) \leq (s - t)|T|.$$

It is clear from the definition that  $B(s, t)$  is a uniformly discrete  $\mathcal{I}_G^*$ -invariant  $G$ -connected Borel instance over  $E_G$ . We will show that there is a constant  $t \in \mathbb{N}$  such that for all  $s \geq t$ ,  $B(s, t)$  is  $\varepsilon$ -correct.

Let

$$d := \begin{cases} |S \cup S^{-1}| & \text{if } S \text{ is finite;} \\ 2 & \text{if } S \text{ is infinite.} \end{cases}$$

**Lemma 6.9.** *For all  $n \in \mathbb{N}$ ,  $\gamma \in \Phi_n$ , and  $x \in X$ , the number of isomorphic embeddings  $\varphi: \Phi_n \rightarrow X$  of  $G_n$  into  $G$  such that  $\varphi(\gamma) = x$  is at most  $d^{|\Phi_n|}$ .*

*Proof.* Notice that if  $S$  is finite, then the maximum degree of  $G$  is at most  $|S \cup S^{-1}| = d$ ; and if  $S$  is infinite, then for any given  $\delta \in S$  and any  $y \in X$ , there are at most 2 edges in  $G$  labeled by  $\delta$  that are incident to  $y$ . Now the statement follows from the connectedness of  $G_n$ . ■

**Lemma 6.10.** *For all  $T \in [E_G]^{<\infty}$ , if  $|\Phi_n| = |T|$ , then there are at most  $|T|d^{|T|}$  isomorphisms between  $G_n$  and  $G|T|$ .*

*Proof.* This is an immediate consequence of Lemma 6.9; indeed, for any  $\gamma \in \Phi_n$ , there are  $|T|$  choices for the image of  $\gamma$  in  $T$ , and any fixed choice can be extended in at most  $d^{|T|}$  ways. ■

**Lemma 6.11.** *For all  $T \in [E_G]^{<\infty}$  and  $k \in \mathbb{N}$ , there are at most  $|T|kd^k$  elements of  $\text{dom}(B(s, t))$  of size  $k$  that intersect  $T$ .*

*Proof.* If there is no  $n \in \mathbb{N}$  with  $|\Phi_n| = k$ , then no element of  $\text{dom}(B(s, t))$  has size  $k$ , so we are trivially done. Suppose that  $|\Phi_n| = k$  (there is a unique such  $n$  since the sequence  $(\Phi_n)_{n=1}^\infty$  is strictly increasing). If  $T' \in \text{dom}(B(s, t))$  and  $|T'| = k$ , then there is an isomorphism  $\varphi: \Phi_n \rightarrow T'$  between  $G_n$  and  $G|T'|$ . If  $T' \cap T \neq \emptyset$ , then there exist  $\gamma \in \Phi_n$  and  $x \in T$  such that  $\varphi(\gamma) = x$ . Now we have  $k$  choices for  $\gamma$ ,  $|T|$  choices for  $x$ , and, by Lemma 6.9, at most  $d^k$  choices for  $\varphi$  given  $\gamma$  and  $x$ . ■

Let  $\nu_s$  denote the uniform probability measure on  $\{0, 1\}^s$ . By Proposition 6.7 and Lemma 6.10, if  $T \in [E_G]^{<\infty}$  and  $|T| = n$ , then

$$\nu_s^T(B_T(s, t)) < nd^n 2^{-tn+1} \leq d^n 2^{-(t-1)n}.$$

Hence, due to Lemma 6.11, to show that  $B(s, t)$  is  $\varepsilon$ -correct, it is enough to find a sequence of numbers  $(p_n)_{n=1}^\infty$ , where each  $p_n \in [0; 1)$ , such that for all  $n \in \mathbb{N}$ ,

$$\varepsilon^{-n} d^n 2^{-(t-1)n} \leq \frac{p_n}{1 - p_n} \prod_{k=1}^\infty (1 - p_k)^{nkd^k}. \quad (6.8)$$

Note that inequality (6.8) does not mention  $s$ ; in other words, if we can solve it, then for any  $s \geq t$ , the instance  $B(s, t)$  will be  $\varepsilon$ -correct.

To solve (6.8), let  $\alpha \in (0; 1)$  be a sufficiently small real number and let  $p_n := \alpha^n$ . Then for every  $n \in \mathbb{N}$ , we have

$$1 - p_n > e^{-2p_n}.$$

Moreover, we can choose  $\alpha$  so that the series

$$\sum_{k=1}^\infty p_k k d^k = \sum_{k=1}^\infty k (\alpha d)^k$$

converges; denote its sum by  $c$ . Now we have

$$\prod_{k=1}^\infty (1 - p_k)^{nkd^k} > e^{-2cn},$$

so (6.8) holds whenever

$$\varepsilon^{-n} d^n 2^{-(t-1)n} \leq \alpha^n e^{-2cn},$$

i.e.,

$$2^{-(t-1)} \leq \alpha e^{-2c} \varepsilon d^{-1}. \quad (6.9)$$

Choose any  $t \in \mathbb{N}$  that satisfies (6.9); for all  $s \geq t$ ,  $B(s, t)$  is  $\varepsilon$ -correct.

If  $B(s, t)$  is  $\varepsilon$ -correct, then it is  $(\varepsilon, G)$ -nice, so there must exist a Borel map  $f: X \rightarrow \{0, 1\}^s$  with  $\mu(D(f, B)) < 1$ . Since  $\mu$  is ergodic, the set  $[X \setminus D(f, B)]_{E_G}$  is  $\mu$ -conull. We claim that for all  $x \in [X \setminus D(f, B)]_{E_G}$ ,

$$\liminf_{m \rightarrow \infty} \frac{K_{\mathbb{O}}(f_m(x))}{|\Phi_m|} \geq s - t.$$

Indeed, let  $x \in X$  and let  $y \in X \setminus D(f, B)$  be such that  $x E_G y$ . Since the sequence  $(\Phi_n)_{n \in \mathbb{N}}$  is increasing and exhaustive, there is  $m_0 \in \mathbb{N}$  such that for all  $m \geq m_0$ ,  $y \in \Phi_m \cdot x$ . This means that for all  $m \geq m_0$ ,  $f|(\Phi_m \cdot x) \notin B(s, t)$ , which, by the definition of  $B(s, t)$ , implies  $K_{\mathbb{O}}(f_m(x)) = K_{\mathbb{O}}(f^{\Phi_m}(x)) > (s - t)|\Phi_m|$ , as desired.

Using Fatou's lemma together with Lemma 6.8, we obtain

$$\begin{aligned} s - t &\leq \int_X \liminf_{m \rightarrow \infty} \frac{K_{\mathbb{O}}(f_m(x))}{|\Phi_m|} d\mu(x) \\ &\leq \liminf_{m \rightarrow \infty} \int_X \frac{K_{\mathbb{O}}(f_m(x))}{|\Phi_m|} d\mu(x) \\ &\leq \limsup_{m \rightarrow \infty} \int_X \frac{K_{\mathbb{O}}(f_m(x))}{|\Phi_m|} d\mu(x) \leq H_{\mu}(a, f). \end{aligned} \tag{6.10}$$

Since for any  $s \geq t$ , we can find  $f$  such that (6.10) holds,  $H_{\mu}(a) = \infty$ , as desired.

**Acknowledgments** This work is supported by the Illinois Distinguished Fellowship. I am grateful to Andrew Zucker for pointing out a mistake in an earlier version of this manuscript and to Anush Tserunyan for introducing me to the field of Borel combinatorics and for her constant support and encouragement.

## References

- [1] N. Alon, J. Grytczuk, M. Hałuszczak, and O. Riordan. *Nonrepetitive colorings of graphs*. Random Structures & Algorithms, Volume 21, Issue 3–4, 2002. Pages 336–346.
- [2] N. Alon, C. McDiarmid, and B. Reed. *Acyclic coloring of graphs*. Random structures and algorithms, Volume 2, No. 3, 1991. Pages 277–288.
- [3] N. Alon, J.H. Spencer. *The Probabilistic Method*. Wiley, New York, 1992.
- [4] J.A. Bondy, U.S.R. Murty. *Graph Theory*. Springer-Verlag, London, 2008.
- [5] O. Borodin, A. Kostochka. *On an upper bound on a graph's chromatic number, depending on the graphs's degree and density*. J. Combin. Theory Ser. B, Volume 23, 1977. Pages 247–250.
- [6] L. Bowen. *Weak isomorphisms between Bernoulli shifts*. Israel Journal of Mathematics, Volume 183, 2011. Pages 93–102.
- [7] A.A. Brudno. *Entropy and the complexity of the trajectories of a dynamical system* (in Russian). Tr. Mosk. Mat. Obs., Volume 44, 1982. Pages 124–149.



- [8] P. Catlin. *Hajos' graph colouring conjecture: variations and counterexamples*. J. Combin. Theory Ser. B, Volume 26, 1979. Pages 268–274.
- [9] C.T. Conley, A.S. Kechris. *Measurable chromatic and independence numbers for ergodic graphs and group actions*. Groups, Geometry and Dynamics, Volume 7, 2013. Pages 127–180.
- [10] C.T. Conley, A.S. Marks, and R. Tucker-Drob. *Brooks's theorem for measurable colorings*. [arXiv:1601.03361](#), preprint, 2016.
- [11] C.T. Conley, B.D. Miller. *A bound on measurable chromatic numbers of locally finite Borel graphs*. Preprint ([link](#)), 2016.
- [12] E. Csóka, Ł. Grabowski, A. Máthé, O. Pikhurko, and K. Tyros. *Borel version of the Local Lemma*. [arXiv:1605.04877](#), preprint, 2016.
- [13] E. Csóka, G. Lippner, and O. Pikhurko. *Kőnig's line coloring and Vizing's theorems for graphings*. [arXiv:1408.1973](#), preprint, 2015.
- [14] P. Erdős, L. Lovász. *Problems and results on 3-chromatic hypergraphs and some related questions*. Infinite and finite sets, A. Hajnal, R. Rado, and V.T. Sós, editors, Colloq. Math. Soc. J. Bolyai, North Holland, 1975. Pages 609–627.
- [15] P. Erdős, A.L. Rubin and H. Taylor. *Choosability in graphs*. Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium XXVI, 1979. Pages 125–157.
- [16] A.M. Frieze, T. Łuczak. *On the independence and chromatic numbers of random regular graphs*. J. Combin. Theory Ser. B, Volume 54, 1992. Pages 123–132.
- [17] T. Jensen, B. Toft. *Graph Coloring Problems*. Wiley, New York, 1995.
- [18] A. Johansson. *Asymptotic choice number for triangle free graphs*. Technical Report 91–95, DIMACS, 1996.
- [19] J. Kahn. *Asymptotics of the list chromatic index for multigraphs*. Random Structures & Algorithms, Volume 17, Issue 2, 2000. Pages 117–156.
- [20] A.S. Kechris. *Classical Descriptive Set Theory*. Springer-Verlag, New York, 1995.
- [21] A.S. Kechris, A. Marks. *Descriptive graph combinatorics*. Preprint ([link](#)), 2016.
- [22] A.S. Kechris, B.D. Miller. *Topics in Orbit Equivalence*. Springer-Verlag, Berlin, Heidelberg, 2004.
- [23] A.S. Kechris, S. Solecki, and S. Todorcevic. *Borel chromatic numbers*. Advances in Mathematics, Volume 141, 1999. Pages 1–44.
- [24] J.H. Kim. *On Brooks' Theorem for sparse graphs*. Combinatorics, Probability and Computing, Volume 4, 1995. Pages 97–132.
- [25] K. Kolipaka, M. Szegedy. *Moser and Tardos meet Lovász*. STOC '11 Proceedings of the forty-third annual ACM symposium on Theory of computing, 2011. Pages 235–244.
- [26] G. Kun. *Expanders have a spanning Lipschitz subgraph with large girth*. [arXiv:1303.4982](#), preprint, 2013.

- [27] J. Lawrence. *Covering the vertex set of a graph with subgraphs of smaller degree*. Discrete Math., Volume 21, 1978. Pages 61–68.
- [28] M. Li, P. Vitányi. *An Introduction to Kolmogorov Complexity and Its Applications*. Springer, New York, 2008.
- [29] R. Lyons, F. Nazarov. *Perfect matchings as IID factors on non-amenable groups*. Europ. J. Combin., Volume 32, 2011. Pages 1115–1125.
- [30] A. Marks. *A determinacy approach to Borel combinatorics*. [arXiv:1304.3830](#), preprint, 2015.
- [31] M. Molloy, B. Reed. *Graph colouring and the probabilistic method*. Algorithms and Combinatorics, Volume 23, Springer-Verlag, Berlin, 2002.
- [32] N. Moriakov. *Computable Følner monotilings and a theorem of Brudno II*. [arXiv:1510.03833](#), preprint, 2015.
- [33] R. Moser, G. Tardos. *A constructive proof of the general Lovász Local Lemma*. J. ACM, Volume 57, Issue 2, 2010.
- [34] D.S. Ornstein, B. Weiss. *Entropy and isomorphism theorems for actions of amenable groups*. Journal d’Analyse Mathématique, Volume 48, 1987. Pages 1–141.
- [35] J. Radhakrishnan, A. Srinivasan. *Improved bounds and algorithms for hypergraph two-coloring*, Random Structures and Algorithms, Volume 16, 2000. Pages 4–32.
- [36] G. Szekeres, H.S. Wilf. *An inequality for the chromatic number of a graph*. Journal of Combinatorial Theory, Volume 4, 1968. Pages 1–3.
- [37] A. Thue. *Über unendliche zeichenreichen* (in German). Norske Vid. Selsk. Skr., I Mat. Nat. Kl., 1906. Pages 1–22.
- [38] A. Tserunyan. *Introduction to Descriptive Set Theory*. Lecture notes ([link](#)), 2016.
- [39] V.A. Uspensky, N.K. Vereshagin, and A. Shen. *Kolmogorov Complexity and Algorithmic Randomness* (in Russian). MCCME, Moscow, 2010.
- [40] V.G. Vizing. *Vertex colorings with given colors* (in Russian). Diskret. Analiz., Volume 29, 1976. Pages 3–10.
- [41] T. Ward, Q. Zhang. *The Abramov–Rokhlin entropy addition formula for amenable group actions*. Monatshefte für Mathematik, Volume 114, 1992. Pages 317–329.
- [42] D. Zheng, E. Chen, and J. Yang. *On large deviations for amenable group actions*. [arXiv:1507.05130](#), preprint, 2016.

## A Proof of Theorem 2.5

For completeness, we present here a proof of Theorem 2.5. Recall that for this theorem we fix a set  $X$  and a correct instance  $B$  over  $X$ . The theorem reads:

**Theorem 2.5.** *Let  $S \in \text{dom}(B)$ . Suppose that  $p: \text{dom}(B) \rightarrow [0; 1]$  is a function witnessing the correctness of  $B$ . Then*

$$\sum_{\mathcal{P} \in \mathfrak{P}(S)} \lambda^{\text{supp}(\mathcal{P}) \times \mathbb{N}}(W(\mathcal{P})) \leq \frac{p(S)}{1 - p(S)}.$$

To prove Theorem 2.5, we establish a correspondence between neat piles and trees of a certain kind. First, let us give some general definitions. For a set  $A$ ,  $A^{<\infty}$  denotes the set of all nonempty finite sequences of elements of  $A$ . For  $s \in A^{<\infty}$ ,  $|s|$  denotes the length of  $s$ , so  $s =: (s_0, \dots, s_{|s|-1})$ . Let  $\ell(s) := s_{|s|-1}$  denote the last element of  $s$ . For  $s, t \in A^{<\infty}$ , we say that  $s$  is a *prefix* of  $t$  (or an *initial segment* of  $t$ ; notation:  $s \subseteq t$ ) if either  $s = t$ , or else, there is a sequence  $u \in A^{<\infty}$  such that  $t = s \hat{\ } u$  (where  $\hat{\ }$  denotes concatenation of finite sequences). A *tree* over  $A$  is a subset  $\mathcal{T} \subseteq A^{<\infty}$  that is (a) closed under taking nonempty initial segments; and (b) contains a unique sequence of length 1, called the *root* of  $\mathcal{T}$ .

If  $<$  is a linear order on  $A$ , then the corresponding *lexicographical order*  $<_{\text{lex}}$  on  $A^{<\infty}$  is defined in the usual way, i.e., for distinct  $s, t \in A^{<\infty}$ ,  $s <_{\text{lex}} t$  if and only if either  $s \subset t$ , or else,  $t \not\subseteq s$  and for the least  $k$  such that  $s_k \neq t_k$ , we have  $s_k < t_k$ . We will use the following properties of the lexicographical order:

1. if  $s \subset t$ , then  $s <_{\text{lex}} t$ ;
2. if  $s <_{\text{lex}} t$  and  $s \not\subseteq t$ , then  $s \hat{\ } u <_{\text{lex}} t \hat{\ } u$  for all  $u \in A^{<\infty}$ .

Now we return to our problem. Fix an arbitrary linear order  $<$  on  $\text{dom}(B)$  and the induced lexicographical order  $<_{\text{lex}}$  on  $\text{dom}(B)^{<\infty}$ . Consider a neat pile  $\mathcal{P}$  with a unique top element  $\tau_0$ . For  $\tau \in \mathcal{P}$ , a  $\tau$ -*path* in  $\mathcal{P}$  is a sequence  $\tau_0, \tau_1, \dots, \tau_k$  of elements of  $\mathcal{P}$  such that  $\tau_k = \tau$  and  $\tau_k \prec \dots \prec \tau_1 \prec \tau_0$ . Note that if  $\tau, \tau', \tau'' \in \mathcal{P}$ ,  $\tau' \prec \tau$ ,  $\tau'' \prec \tau$ , and  $\text{dom}(\tau') = \text{dom}(\tau'')$ , then  $\tau' = \tau''$ . Indeed, suppose that  $\tau' \neq \tau''$  and let  $T := \text{dom}(\tau') = \text{dom}(\tau'')$ . By the definition of the relation  $\prec$ , there exist  $x_1, x_2 \in T$  such that  $\tau'(x_1) = \tau(x_1) - 1$  and  $\tau''(x_2) = \tau(x_2) - 1$ . Since  $\mathcal{P}$  is neat,  $\tau'(x_2) < \tau(x_2)$ ; since the graphs of  $\tau'$  and  $\tau''$  are disjoint, we get  $\tau'(x_2) < \tau''(x_2)$ . Similarly, we have  $\tau''(x_1) < \tau'(x_1)$ , which contradicts the neatness of  $\mathcal{P}$ .

From the above observation it follows that for every  $u \in \text{dom}(B)^{<\infty}$ , there is at most one path  $\tau_0, \dots, \tau_{|u|-1}$  in  $\mathcal{P}$  such that  $u_i = \text{dom}(\tau_i)$  for all  $0 \leq i \leq |u| - 1$ ; if such a path exists, then we denote the element  $\tau_{|u|-1}$  in it by  $\tau_{\mathcal{P}}(u)$ . Note that if  $v \subset u$  and  $\tau_{\mathcal{P}}(u)$  is defined, then  $\tau_{\mathcal{P}}(v)$  is also defined and  $\tau_{\mathcal{P}}(v) \neq \tau_{\mathcal{P}}(u)$ . For  $\tau \in \mathcal{P}$ , let  $u_{\mathcal{P}}(\tau)$  denote the lexicographically largest sequence  $u \in \text{dom}(B)^{<\infty}$  such that  $\tau_{\mathcal{P}}(u) = \tau$  (note that the set of such sequences is nonempty and finite, so this definition always makes sense).

**Lemma A.1.** *Let  $\mathcal{P}$  be a neat pile with a unique top element. Suppose that  $\tau, \tau' \in \mathcal{P}$  and  $x \in \text{dom}(\tau) \cap \text{dom}(\tau')$ . Then*

$$\tau'(x) < \tau(x) \iff u_{\mathcal{P}}(\tau) <_{\text{lex}} u_{\mathcal{P}}(\tau').$$

*Proof.* Let  $u := u_{\mathcal{P}}(\tau)$  and let  $u' := u_{\mathcal{P}}(\tau')$ . If, say,  $\tau'(x) < \tau(x)$ , then there is a sequence  $\tau_1, \dots, \tau_k$  of elements of  $\mathcal{P}$  such that  $\tau_1 = \tau$ ,  $\tau_k = \tau'$ , and  $\tau_k \prec \dots \prec \tau_1$ . Therefore, the sequence  $v := u \hat{\ } (\text{dom}(\tau_2), \dots, \text{dom}(\tau_k))$  satisfies  $\tau_{\mathcal{P}}(v) = \tau'$ . Hence,  $u <_{\text{lex}} v \leq_{\text{lex}} u'$ , as desired. The case  $\tau(x) < \tau'(x)$  is the same, *mutatis mutandis*. ■

Call a sequence  $u \in \text{dom}(B)^{<\infty}$  *proper* if for all  $0 \leq i < |u| - 1$ ,  $u_i \cap u_{i+1} \neq \emptyset$ . Note that if  $\mathcal{P}$  is a neat pile with a unique top element, then  $\tau_{\mathcal{P}}(u)$  can only be defined if  $u$  is proper. A tree  $\mathcal{T}$  over  $\text{dom}(B)$  is *proper* if every element of  $\mathcal{T}$  is proper. For  $S \in \text{dom}(B)$ , let  $\mathfrak{T}(S)$  denote the set of all proper finite trees with root  $(S)$ .

**Lemma A.2.** *Let  $S \in \text{dom}(B)$  and let  $\mathcal{P} \in \mathfrak{P}(S)$ . Then*

$$\mathcal{T}_{\mathcal{P}} := \{u_{\mathcal{P}}(\tau) : \tau \in \mathcal{P}\} \in \mathfrak{T}(S).$$

*Moreover, the map  $\mathfrak{P}(S) \rightarrow \mathfrak{T}(S): \mathcal{P} \mapsto \mathcal{T}_{\mathcal{P}}$  is injective.*

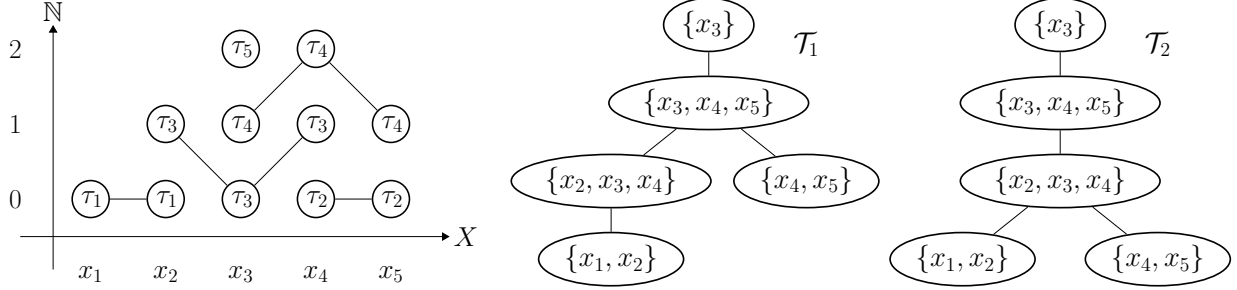


Figure 3: A neat pile  $\mathcal{P} = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$  with unique top element  $\tau_5$ . If  $\{x_2, x_3, x_4\} < \{x_4, x_5\}$ , then  $\mathcal{T}_{\mathcal{P}} = \mathcal{T}_1$ ; otherwise,  $\mathcal{T}_{\mathcal{P}} = \mathcal{T}_2$ .

*Proof.* Let  $\mathcal{P} \in \mathfrak{P}(S)$  and let  $\mathcal{T} := \mathcal{T}_{\mathcal{P}}$ . Denote the unique top element of  $\mathcal{P}$  by  $\tau_0$ ; note that  $\text{dom}(\tau_0) = S$ . Suppose that  $u \in \mathcal{T}$  and  $v \in \text{dom}(B)^{<\infty}$  is such that  $v \subset u$ . We claim that  $u_{\mathcal{P}}(\tau_{\mathcal{P}}(v)) = v$  (and thus  $v \in \mathcal{T}$ ). Indeed, suppose that  $u = v \wedge w$ , where  $w \in \text{dom}(B)^{<\infty}$ . If  $u_{\mathcal{P}}(\tau_{\mathcal{P}}(v)) \neq v$ , then  $v <_{\text{lex}} u_{\mathcal{P}}(\tau_{\mathcal{P}}(v))$  and  $v \not\subset u_{\mathcal{P}}(\tau_{\mathcal{P}}(v))$ , so  $u = v \wedge w <_{\text{lex}} u_{\mathcal{P}}(\tau_{\mathcal{P}}(v)) \wedge w$ . However,  $\tau_{\mathcal{P}}(u_{\mathcal{P}}(\tau_{\mathcal{P}}(v)) \wedge w) = \tau_{\mathcal{P}}(u)$ , which contradicts the choice of  $u$ . Therefore,  $\mathcal{T}$  is closed under taking nonempty initial segments. The rest of the proof that  $\mathcal{T} \in \mathfrak{T}(S)$  is straightforward.

To see that the map  $\mathcal{P} \mapsto \mathcal{T}_{\mathcal{P}}$  is injective, consider any  $\tau \in \mathcal{P}$  and  $x \in \text{dom}(\tau)$ . It is easy to see that

$$\tau(x) = |\{\tau' \in \mathcal{P} : x \in \text{dom}(\tau') \text{ and } \tau'(x) < \tau(x)\}|.$$

Therefore, by Lemma A.1, for any  $u \in \mathcal{T}$  and  $x \in \ell(u)$ ,

$$\begin{aligned} \tau_{\mathcal{P}}(u)(x) &= |\{\tau' \in \mathcal{P} : x \in \text{dom}(\tau') \text{ and } \tau'(x) < \tau_{\mathcal{P}}(u)(x)\}| \\ &= |\{u' \in \mathcal{T} : x \in \ell(u') \text{ and } u <_{\text{lex}} u'\}|. \end{aligned}$$

The last expression only depends on  $\mathcal{T}$ ; hence,  $\mathcal{P}$  can be recovered from  $\mathcal{T}$ , as desired.  $\blacksquare$

Recall that for  $\mathcal{P} \in \mathfrak{P}(S)$ ,

$$\lambda^{\text{supp}(\mathcal{P}) \times \mathbb{N}}(W(\mathcal{P})) = \prod_{\tau \in \mathcal{P}} \lambda^{\text{dom}(\tau)}(B_{\text{dom}(\tau)}) = \prod_{u \in \mathcal{T}_{\mathcal{P}}} \lambda^{\ell(u)}(B_{\ell(u)}).$$

Hence, due to Lemma A.2,

$$\sum_{\mathcal{P} \in \mathfrak{P}(S)} \lambda^{\text{supp}(\mathcal{P}) \times \mathbb{N}}(W(\mathcal{P})) \leq \sum_{\mathcal{T} \in \mathfrak{T}(S)} \prod_{u \in \mathcal{T}} \lambda^{\ell(u)}(B_{\ell(u)}).$$

Theorem 2.5 now follows from the following lemma:

**Lemma A.3.** *Let  $S \in \text{dom}(B)$ . Suppose that  $p: \text{dom}(B) \rightarrow [0; 1]$  is a function witnessing the correctness of  $B$ . Then*

$$\sum_{\mathcal{T} \in \mathfrak{T}(S)} \prod_{u \in \mathcal{T}} \lambda^{\ell(u)}(B_{\ell(u)}) \leq \frac{p(S)}{1 - p(S)}.$$

*Proof.* For a finite tree  $\mathcal{T}$  over  $\text{dom}(B)$ , let the *weight* of  $\mathcal{T}$  be

$$\omega(\mathcal{T}) := \prod_{u \in \mathcal{T}} \lambda^{\ell(u)}(B_{\ell(u)}).$$

For  $n \geq 1$ , let  $\mathfrak{T}_{\leq n}(S)$  denote the subset of  $\mathfrak{T}(S)$  consisting of all trees of height at most  $n$  (where the *height* of a tree  $\mathcal{T}$  is the maximum of the lengths of its elements). Note that  $\mathfrak{T}(S) = \bigcup_{n=1}^{\infty} \mathfrak{T}_{\leq n}(S)$  and the union is increasing, so

$$\sum_{\mathcal{T} \in \mathfrak{T}(S)} \omega(\mathcal{T}) = \lim_{n \rightarrow \infty} \sum_{\mathcal{T} \in \mathfrak{T}_{\leq n}(S)} \omega(\mathcal{T}).$$

Therefore, it is enough to show that for all  $n \geq 1$ ,

$$\sum_{\mathcal{T} \in \mathfrak{T}_{\leq n}(S)} \omega(\mathcal{T}) \leq \frac{p(S)}{1 - p(S)}. \quad (\text{A.1})$$

We prove (A.1) by induction on  $n$ . Note that the only tree in  $\mathfrak{T}_{\leq 1}(S)$  is  $\{(S)\}$ , and

$$\omega(\{(S)\}) = \lambda^S(B_S) \leq \frac{p(S)}{1 - p(S)}$$

by (1.3). Now suppose that (A.1) holds for some  $n$ . Consider any  $\mathcal{T} \in \mathfrak{T}_{\leq n+1}(S)$ . Note that if  $u \in \mathcal{T}$ , then either  $u = (S)$ , or else,  $u = (S)^\frown v$  for some  $v \in \text{dom}(B)^{<\infty}$  such that  $v_0 \cap S \neq \emptyset$ . For each  $S' \in \text{dom}(B)$  with  $S' \cap S \neq \emptyset$ , define

$$\mathcal{T}_{S'} := \{v \in \text{dom}(B)^{<\infty} : (S)^\frown v \in \mathcal{T} \text{ and } v_0 = S'\}.$$

In other words, if  $(S, S') \in \mathcal{T}$ , then  $\mathcal{T}_{S'}$  is the subtree of  $\mathcal{T}$  rooted at  $(S, S')$ ; and otherwise  $\mathcal{T}_{S'} = \emptyset$ . This defines a bijection between  $\mathfrak{T}_{\leq n+1}(S)$  and the set

$$\prod_{\substack{S' \in \text{dom}(B): \\ S' \cap S \neq \emptyset}} (\{\emptyset\} \cup \mathfrak{T}_{\leq n}(S')).$$

Moreover, if we set  $\omega(\emptyset) := 1$ , then

$$\omega(\mathcal{T}) = \lambda^S(B_S) \prod_{\substack{S' \in \text{dom}(B): \\ S' \cap S \neq \emptyset}} \omega(\mathcal{T}_{S'}).$$

Now we have

$$\begin{aligned}
\sum_{\mathcal{T} \in \mathfrak{T}_{\leq n+1}(S)} \omega(\mathcal{T}) &= \sum_{\mathcal{T} \in \mathfrak{T}_{\leq n+1}(S)} \lambda^S(B_S) \prod_{\substack{S' \in \text{dom}(B): \\ S' \cap S \neq \emptyset}} \omega(\mathcal{T}_{S'}) \\
&= \lambda^S(B_S) \prod_{\substack{S' \in \text{dom}(B): \\ S' \cap S \neq \emptyset}} \left( 1 + \sum_{\mathcal{T} \in \mathfrak{T}_{\leq n}(S')} \omega(\mathcal{T}) \right) \\
[\text{by the induction hypothesis}] &\leq \lambda^S(B_S) \prod_{\substack{S' \in \text{dom}(B): \\ S' \cap S \neq \emptyset}} \left( 1 + \frac{p(S')}{1 - p(S')} \right) \\
&= \lambda^S(B_S) \prod_{\substack{S' \in \text{dom}(B): \\ S' \cap S \neq \emptyset}} \frac{1}{1 - p(S')} \\
[\text{by (1.3)}] &\leq \frac{p(S)}{1 - p(S)},
\end{aligned}$$

as desired. ■

## B Theorems of Kim, Johansson, and Kahn

Theorems 4.1 and 5.4 are designed in such a way that the proofs of many classical combinatorial results can be transferred to the measurable setting simply by replacing the LLL with Theorem 4.1 or 5.4, depending on the desired outcome. Almost no additional work is required; the relevant instances of the LLL and the verification of their correctness remain unchanged, so the only things to check are the Borelness of the instances and their hereditary local finiteness (for Theorem 4.1) or invariance (for Theorem 5.4), which is usually straightforward. In the introduction we mentioned Theorems 1.1, 1.3, 1.4, and 1.5 as particular examples of this approach; they are obtained by substituting Theorems 4.1 and 5.4 in the proofs of Theorems 1.11, 1.12, and 1.14.

The proofs of Theorems 1.11, 1.12, and 1.14 themselves are quite technical and somewhat lengthy (and, arguably, constitute some of the finest and most sophisticated known applications of the LLL); the relevant instances of the LLL are carefully engineered to ensure their correctness and yet to give their solutions sufficiently strong properties. An excellent exposition of all three proofs, along with the intuition behind them, can be found in [31], and we do not attempt to reproduce it here. However, for the interested reader, we very briefly sketch in this appendix the main ideas of the method used to prove Theorems 1.11, 1.12, and 1.14, omitting most of the technical details.

Consider a graph  $G$  on a set  $X$  with maximum degree  $d \in \mathbb{N}$ . A useful, although not strictly necessary, observation (see [31, Section 1.5]) is that we can usually assume  $G$  to be  $d$ -regular by attaching to each vertex  $x \in X$  with  $\deg_G(x) < d$  an infinite rooted tree whose root has degree  $d - \deg_G(x)$  and all of whose other vertices have degree  $d$ .<sup>14</sup> Formally, let  $Y$

---

<sup>14</sup>This method of making a graph regular is different from the one described in [31, Section 1.5], where the attention is restricted to finite graphs only.

be the set of all pairs of the form  $(x, s)$ , where  $x \in X$  and  $s = (s_1, \dots, s_n)$  is a finite sequence of integers such that  $1 \leq s_1 \leq d - \deg_G(x)$  and  $1 \leq s_i \leq d - 1$  for all  $1 < i \leq n$  (with the case  $n = 0$  and  $s = \emptyset$  allowed). Define a graph  $H$  on  $Y$  as follows:

$$(x, s)H(y, t) :\iff (xGy \text{ and } s = t = \emptyset) \text{ or } (x = y \text{ and } (s \subset t \text{ or } t \subset s)).$$

Clearly,  $H$  is  $d$ -regular and the map  $x \mapsto (x, \emptyset)$  is an isomorphic embedding of  $G$  into  $H$  (denote its image by  $G^*$ ). Any cycle in  $H$  has to be contained in  $G^*$ , so  $g(H) = g(G^*) = g(G)$ . Finally, note that if  $G$  is a Borel graph on a standard Borel space  $X$ , then  $Y$  and  $H$  are Borel subsets of  $\text{HF}_0(E_G)$ ,  $H$  is a Borel graph on  $Y$ , and every isomorphism between connected components of  $G$  extends naturally to an isomorphism between connected components of  $H$ .

To illustrate the technique employed in the proofs of Theorems 1.11, 1.12, and 1.14, we will outline here a proof of the following weakening of Johansson's theorem: *There exists a positive constant  $\varepsilon$  such that any triangle-free graph  $G$  with maximum degree  $d \in \mathbb{N}$  satisfies  $\chi(G) \leq (1 - \varepsilon)d + o(d)$*  (see [31, Theorem 10.2]).

Fix  $\varepsilon > 0$ , and let  $G$  be a  $d$ -regular triangle-free graph on a set  $X$ , where  $d$  is assumed to be sufficiently large. The argument will proceed in two steps. First, we will apply the LLL to produce a partial coloring of  $G$  that exhibits the same local behavior as a “typical” random coloring. On the second step, we extend this partial coloring to a full coloring of  $G$  using the fact that the uncolored part of the graph is “space.” Let us start by explaining how the second step works. Suppose that we are given a subset  $X' \subseteq X$  and a proper coloring  $f: X' \rightarrow \mathbb{N}$  of  $G|X'$ . We can extend  $f$  to a proper coloring of the whole graph  $G$  in the following “greedy” way. Fix a proper coloring  $c: X \setminus X' \rightarrow \mathbb{N}$  of  $G|(X \setminus X')$  (which exists since  $G$  is locally finite). Set  $f_0 := f$  and for all  $n \in \mathbb{N}$ , define  $f_{n+1}: \text{dom}(f_n) \cup c^{-1}(n) \rightarrow \mathbb{N}$  as follows:

$$f_{n+1}(x) := \begin{cases} f_n(x) & \text{if } x \in \text{dom}(f_n); \\ \min\{i \in \mathbb{N} : \neg \exists y \in G_x \cap \text{dom}(f_n)(f_n(y) = i)\} & \text{if } c(x) = n. \end{cases}$$

Finally, set  $f_\infty := \bigcup_{n=0}^\infty f_n$ . By construction,  $f_\infty$  is a proper coloring of  $G$ . How many colors does it use? Note that for every  $n \in \mathbb{N}$  and  $x \in c^{-1}(n)$ , there can be at most  $d$  distinct colors assigned by  $f_n$  to the neighbors of  $x$ , and so  $f_{n+1}(x) \leq d$ . Hence, for all  $x \in X \setminus X'$ ,  $f_\infty(x) \leq d$ , i.e.,  $f_\infty|(X \setminus X')$  uses at most  $d + 1$  colors.

The above upper bound on  $f_{n+1}(x)$  for  $x \in c^{-1}(n)$  is sharp only if all the neighbors of  $x$  are given different colors by  $f_n$ . This observation motivates the following definition: A partial proper coloring  $f: X' \rightarrow \mathbb{N}$  is *good* if for every  $x \in X \setminus X'$ , the following set has cardinality at least  $\varepsilon d$ :

$$\{i \in \mathbb{N} : |\{y \in G_x \cap X' : f(y) = i\}| \geq 2\}.$$

If  $f$  is good, then clearly  $f_\infty(x) \leq (1 - \varepsilon)d$  for all  $x \in X \setminus X'$ . Thus, to finish the proof, we only need to find a good partial coloring  $f: X' \rightarrow \{0, \dots, (1 - \varepsilon)d + o(d)\}$ .

To this end, we will apply the LLL. For a function  $f: X \rightarrow \{0, \dots, \lfloor d/2 \rfloor\}$ , let

$$X_f := \{x \in X : \neg \exists y \in G_x (f(x) = f(y))\}.$$

In other words,  $X_f$  is the set of all vertices that do not share a color with any of their neighbors. We want to find a function  $f: X \rightarrow \{0, \dots, \lfloor d/2 \rfloor\}$  with the property that  $f|X_f$

is a good partial coloring (note that the definition of  $X_f$  ensures that this coloring is proper). This condition can be easily turned into an instance of the LLL whose domain consists of all sets of the form

$$G_x \cup \bigcup_{y \in G_x} G_y,$$

where  $x \in X$ . It turns out that for sufficiently small  $\varepsilon$ , this instance is correct, and hence a desired good partial coloring exists; informally, if all the  $d$  neighbors of a vertex  $x \in X$  are colored randomly using only  $\lfloor d/2 \rfloor$  colors, then one should expect many colors to be repeated, and since  $G$  is triangle-free, shared colors between vertices in the neighborhood of  $x$  do not force them to be removed from  $X_f$ . The rigorous verification of this fact constitutes the most laborious part of the argument. For the details, see [31, Theorem 10.2].

The proofs of Theorems 1.11, 1.12, and 1.14 utilize a similar strategy to the argument outlined above but with several clever technical twists. The main difference is that the LLL is applied repeatedly to produce partial colorings of larger and larger subsets of  $X$  (or, in the case of Theorem 1.14, larger and larger subsets of the edge set of  $G$ ). Another difference is that the final step, instead of completing the coloring “greedily,” also uses the LLL, in the form of the following lemma:

**Lemma B.1** ([31, Theorem 4.3]). *Let  $G$  be a locally countable graph with vertex set  $X$ . Let  $k \in \mathbb{N} \setminus \{0\}$ . Suppose that  $L: X \rightarrow [\mathbb{N}]^{<\infty}$  is a function such that for all  $x \in X$ ,  $|L(x)| \geq k$  and for each  $n \in L(x)$ ,*

$$|\{y \in G_x : n \in L(y)\}| \leq k/8.$$

*Then there exists a proper coloring  $f: X \rightarrow \mathbb{N}$  with  $f(x) \in L(x)$  for all  $x \in X$ .*

Lemma B.1 is established using a straightforward application of the LLL. Note that the instance to which the LLL is applied there is only invariant under those isomorphisms between connected components of  $G$  that preserve the value of  $L$  (i.e., it is  $\mathcal{I}_G[L]$ -invariant). In the situations under consideration,  $L$  is defined using the outcomes of the previous applications of the LLL, so Theorem 5.4 applies.

As mentioned before, the precise intricate construction of the instances of the LLL used in the proofs of Theorems 1.11, 1.12, and 1.14 lies outside the scope of this article. However, the general scheme described above already shows that the LLL can be replaced by Theorem 4.1 or Theorem 5.4; the details of the proofs—verifying the correctness of the instances—do not require any modification.